Extension theorems for reductive group schemes

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ABSTRACT. We prove several basic extension theorems for reductive group schemes that have meaningful crystalline applications to the existence of good integral models of Shimura varieties of Hodge type. We also prove that each Lie algebra with a perfect Killing form over a commutative $\mathbb{Z}$-algebra, is the Lie algebra of an adjoint group scheme.

Key words: reductive group schemes, purity, regular rings, Lie algebras, and F-crystals.

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1. Introduction

A group scheme F over a scheme S is called *reductive* if the morphism $F \to S$ has the following two properties: (i) it is smooth and affine (and therefore of finite presentation) and (ii) its fibres are reductive groups over spectra of fields and therefore are connected (cf. [DG, Vol. III, Exp. XIX, 2.7, 2.1, and 2.9]). If the center of F is trivial, then F is called an *adjoint* group scheme over S. Let $\mathcal{O}_Y$ be the structure ring sheaf of a scheme Y. Let U be an open, Zariski dense subscheme of Y. We call the pair $(Y, Y \setminus U)$ *quasi-pure* if each étale cover of U extends uniquely to an étale cover of Y (to be compared with [Gr2, Exp. X, 3.1]). The main goal of this paper is to prove the following Theorem (see Section 4) and to present a new crystalline application of it.

1.1. Basic Theorem A. (a) *Suppose that $Y = Spec(A)$ is an affine scheme. Let K be the ring of fractions of A. Let G_K be an adjoint group scheme over $Spec(K)$ such that the symmetric bilinear Killing form on the Lie algebra $Lie(G_K)$ of G_K is perfect. We assume that there exists a Lie algebra $\mathfrak{g}$ over A such that the following two properties hold:*

(i) *we have an identity $Lie(G_K) = \mathfrak{g} \otimes_A K$ and the A-module $\mathfrak{g}$ is projective and locally has finite rank;*

(ii) *the symmetric bilinear Killing form on $\mathfrak{g}$ is perfect (i.e., it induces naturally an A-linear isomorphism $\mathfrak{g} \xrightarrow{\sim} Hom_A(\mathfrak{g}, A)$).*

Then there exists a unique adjoint group scheme G over Y which extends G_K and such that we have an identity $Lie(G) = \mathfrak{g}$ that extends the identity of (i).

(b) *Suppose that Y is a normal, noetherian scheme and the codimension of $Y \setminus U$ in Y is at least 2. Let G_U be an adjoint group scheme over U. We assume that the Lie*

algebra $\mathcal{O}_U$-module $Lie(G_U)$ of G_U extends to a Lie algebra $\mathcal{O}_Y$-module $\mathfrak{l}$ that is a locally free $\mathcal{O}_Y$-module. Then G_U extends uniquely to an adjoint group scheme G over Y.

(c) *Let Y and U be as in (b). Let E_U be a reductive group scheme over U. We assume that the pair $(Y, Y \setminus U)$ is quasi-pure and that the Lie algebra $\mathcal{O}_U$-module $Lie(G_U)$ of the adjoint group scheme G_U of E_U extends to a Lie algebra $\mathcal{O}_Y$-module $\mathfrak{l}$ that is a locally free $\mathcal{O}_Y$-module. Then E_U extends uniquely to a reductive group scheme E over Y.*

The proof of Theorem 1.1 (a) we include combines the cohomology theory of Lie algebras with a simplified variant of [Va1, Claim 2, p. 464] (see Subsections 3.3, 3.5, and 4.1). The proof of Theorem 1.1 (b) is an application of [CTS, Cor. 6.12] (see Subsection 4.2). The classical purity theorem of Nagata and Zariski (see [Gr2, Exp. X, 3.4 (i)]) says that the pair $(Y, Y \setminus U)$ is quasi-pure, provided Y is regular and U contains all points of Y of codimension 1 in Y. In such a case, a slightly weaker form of Theorem 1.1 (c) was obtained in [CTS, Thm. 6.13]. In general, the hypotheses of Theorem 1.1 (b) and (c) are needed (see Remarks 4.4). See [MB] (resp. [FC], [Va1], and [Va2]) for different analogues of Theorem 1.1 (c) for Jacobian (resp. for abelian) schemes. For instance, in [Va2, Thm. 1.3] we proved that if Y is a regular, formally smooth scheme over the ring of Witt vectors with coefficients in a perfect field l and if U contains all points of Y that are either of characteristic 0 or of codimension 1 in Y, then each abelian scheme over U extends (automatically in a unique way) to an abelian scheme over Y.

In Section 2 we include notations and basic results. In Section 3 we prove that each Lie algebra over A which as an A-module is projective and locally has finite rank and whose Killing form is perfect, is the Lie algebra of an adjoint group scheme over $\mathrm{Spec}(A)$ (see Theorem 3.5). In Section 4 we prove the Basic Theorem A. In Section 5 we include three results on extending homomorphisms between reductive group schemes. The first one is an application of Theorem 1.1 (c) (see Proposition 5.1) and the other two are refinements of results of [Va1] (see Subsections 5.2 to 5.5; in particular, see Subsection 5.3 for the Basic Theorem B). Our main motivation in obtaining the Basic Theorem A stems from the meaningful applications to crystalline cohomology one gets by combining it with either Faltings' results of [Fa, §4] (see [Va1]; see also Subsections 5.3 to 5.5 and 6.4) or with de Jong's extension theorem [dJ, Thm. 1.1] (see Section 6). In Section 6 we formalize such applications which rely on [dJ], in a purely abstract context.

The idea behind the *crystalline reductive extension principle* (i.e., behind the Basic Theorem C of 6.3) can be summarized as follows. Suppose that the perfect field l is algebraically closed. Let $\mathcal{M}$ be a moduli space over $\mathrm{Spec}(l)$ of finite type. Suppose that to each point $\mathcal{X} \in \mathcal{M}(l)$ one can associate a flat, affine group scheme $\mathcal{G}_{\mathcal{X}}$ over $\mathrm{Spec}(W(l))$ such that the following two properties hold:

(i) the moduli space $\mathcal{M}$ is smooth at $\mathcal{X}$ if and only if (the normalization of) $\mathcal{G}_{\mathcal{X}}$ is a reductive group scheme over $\mathrm{Spec}(W(l))$;

(ii) the association $\mathcal{X} \to \mathcal{G}_{\mathcal{X}}$ is defined naturally by an F-crystal over $\mathcal{M}$ endowed with a good family of crystalline tensors.

Then based on the Basic Theorem C one concludes that: *the smooth locus of $\mathcal{M}$ over $\mathrm{Spec}(l)$ is an open closed subscheme of $\mathcal{M}$.*

In a future work we will use Section 6 to extend our work on integral canonical models of Shimura varieties of Hodge type in unramified mixed characteristic $(0,p)$ with p a prime greater than 3 (see [Va1]), to unramified mixed characteristics $(0,2)$ and $(0,3)$. Though Theorem 1.1 (a) and the Basic Theorem B are used in Section 6 only for very simple cases, for the sake of completeness we present these two results in much greater generality: these two results review the very essence of the fundamental result [Va1, Prop. 4.3.10] in a significantly refined and simplified manner and implicitly emphasize the new range of applicability of the Basic Theorem C versus the old range of applicability of [Va1, Subsections 4.3, 5.2, and 5.3] (see Remarks 6.4).

2. Preliminaries

Our notations are gathered in Subsection 2.1. In Subsections 2.2 to 2.6 we include five basic results that are of different nature and are often used in Sections 3 to 6.

2.1. Notations and conventions. If K is a field, let $\bar{K}$ be an algebraic closure of it. Let F be a reductive group scheme over a scheme S. Let $Z(F)$, F^{der}, F^{ad}, and F^{ab} denote the center, the derived group scheme, the adjoint group scheme, and the abelianization of F (respectively). We have $Z^{\mathrm{ab}}(F) = F/F^{\mathrm{der}}$ and $F^{\mathrm{ad}} = F/Z(F)$. The center $Z(F)$ is a group scheme of multiplicative type, cf. [DG, Vol. III, Exp. XXII, Cor. 4.1.7]. Let $Z^0(F)$ be the maximal torus of $Z(F)$; the quotient group scheme $Z(F)/Z^0(F)$ is a finite, flat group scheme over S of multiplicative type. Let F^{sc} be the simply connected semisimple group scheme cover of the derived group scheme F^{der}. See [DG, Vol. III, Exp. XXII, Cor. 4.3.2] for the quotient group scheme F/H of F by a flat, closed subgroup scheme H of $Z(F)$ which is of multiplicative type. If X or X_S is an S-scheme, let X_{A_1} (resp. X_{S_1}) be its pull back via a morphism $\mathrm{Spec}(A_1) \to S$ (resp. $S_1 \to S$). If S is either affine or integral, let K_S be the ring of fractions of S. If S is a normal, noetherian, integral scheme, let $\mathcal{D}(S)$ be the set of local rings of S that are discrete valuation rings. Let $\mathbb{G}_{mS}$ be the rank 1 split torus over S; similarly, the group schemes $\mathbb{G}_{aS}$, GL_{nS}, etc., will be understood to be over S. Let $\mathit{Lie}(F)$ be the Lie algebra $\mathcal{O}_S$-module of F. If $S = \mathrm{Spec}(A)$ is affine, then let $\mathbb{G}_{mA} := \mathbb{G}_{mS}$, etc., and let $\mathrm{Lie}(\tilde{F})$ be the Lie algebra over A of a closed subgroup scheme $\tilde{F}$ of F. As A-modules, we identify $\mathrm{Lie}(\tilde{F}) = \mathrm{Ker}(\tilde{F}(A[x]/x^2) \to \tilde{F}(A))$, where the A-epimorphism $A[x]/(x^2) \twoheadrightarrow A$ takes x to 0. The Lie bracket on $\mathrm{Lie}(\tilde{F})$ is defined by taking the (total) differential of the commutator morphism $[,] : \tilde{F} \times_S \tilde{F} \to \tilde{F}$ at identity sections. If $S = \mathrm{Spec}(A)$ is affine, then $\mathrm{Lie}(F) = \mathit{Lie}(F)(S)$ is the Lie algebra over A of global sections of $\mathit{Lie}(F)$. If $S_1 \to S$ is an étale cover and if F_1 is a reductive group scheme over S_1, let $\mathrm{Res}_{S_1/S} F_1$ be the reductive group scheme over S that is the Weil restriction of scalars of F_1 (see [BLR, Ch. 7, 7.6] and [Va3, Subsection 2.3]). We have a functorial group identity $\mathrm{Res}_{S_1/S} F_1(X) = F_1(X_{S_1})$.

If N is a projective A-module which locally has finite rank, let $N^* := \mathrm{Hom}_A(N, A)$, let GL_N be the reductive group scheme over $\mathrm{Spec}(A)$ of linear automorphisms of N, and let $\mathfrak{gl}_N := \mathrm{Lie}(GL_N)$. Thus $\mathfrak{gl}_N$ is the Lie algebra associated to the A-algebra $\mathrm{End}_A(N)$. In Section 6 we will use the following identifications

$$\mathrm{End}_A(\mathrm{End}_A(N, N)) = \mathrm{End}_A(N \otimes_A N^*)$$

$$= N \otimes_A N^* \otimes_A N^* \otimes_A N = N \otimes_A N \otimes_A N^* \otimes_A N^* = \mathrm{End}_A(N \otimes_A N).$$

A bilinear form $b_N : N \times N \to A$ on N is called perfect if it induces an A-linear map $N \to N^*$ that is an isomorphism. If b_N is symmetric, then by the kernel of b_N we mean the A-submodule $\mathrm{Ker}(b_N) := \{a \in N | b_N(a,b) = 0\ \forall b \in N\}$ of N. For a Lie algebra $\mathfrak{g}$ over A that is a projective A-module which locally has finite rank, let $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$ be the adjoint representation of $\mathfrak{g}$ and let $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to A$ be the Killing form on $\mathfrak{g}$. For a, $b \in \mathfrak{g}$ we have $\mathrm{ad}(a)(b) = [a,b]$ and $\mathcal{K}_{\mathfrak{g}}(a,b)$ is the trace of the endomorphism $\mathrm{ad}(a) \circ \mathrm{ad}(b)$ of $\mathfrak{g}$. The kernel $\mathrm{Ker}(\mathcal{K}_{\mathfrak{g}})$ is an ideal of $\mathfrak{g}$.

We denote by k an arbitrary field. Let $n \in \mathbb{N}^*$. See [Bou2, §4] and [Hu1, §11] for the classification of connected Dynkin diagrams. We say F is of isotypic $\mathcal{L} \in \{A_n, B_n, C_n | n \in \mathbb{N}^*\} \cup \{D_n | n \geq 3\} \cup \{E_6, E_7, E_8, F_4, G_2\}$ Dynkin type if the connected Dynkin diagram of each simple factor of an arbitrary geometric fibre of F^{ad}, is $\mathcal{L}$; if F^{ad} is absolutely simple we drop the word isotypic. We recall that $A_1 = B_1 = C_1$, $B_2 = C_2$, and $A_3 = D_3$.

2.2. Proposition. *Let Y be a normal, noetherian, integral scheme. Let $K := K_Y$.*

(a) *If $Y = Spec(A)$ is affine, then inside K we have $A = \cap_{V \in \mathcal{D}(Y)} V$.*

(b) *Let U be an open subscheme of Y such that $Y \setminus U$ has codimension in Y at least 2. Let W be an affine Y-scheme of finite type. Then the natural restriction map $W(Y) \to W(U)$ is a bijection. If moreover W is integral, normal, dominates Y, and such that we have $\mathcal{D}(W) = \mathcal{D}(W_U)$, then W is determined (up to unique isomorphism) by W_U.*

(c) *Suppose that $Y = Spec(A)$ is local, regular, and has dimension 2. Let y be the closed point of Y and let $U := Y \setminus \{y\}$. Then each locally free $\mathcal{O}_U$-module which locally has finite rank, extends uniquely to a free $\mathcal{O}_Y$-module.*

Proof: See [Ma, (17.H), Thm. 38] for (a). To check (b), we can assume $Y = \mathrm{Spec}(A)$ is affine. We write $W = \mathrm{Spec}(B)$. The A-algebra of global functions of U is A, cf. (a). We have $\mathrm{Hom}_Y(U,W) = \mathrm{Hom}_A(B,A) = \mathrm{Hom}_Y(Y,W)$. If moreover B is a normal ring that contains A and we have $\mathcal{D}(W) = \mathcal{D}(W_U)$, then B is uniquely determined by $\mathcal{D}(W_U)$ (cf. (a)) and therefore by W_U. From this (b) follows. See [Gr2, Exp. X, 3.5] for (c). $\square$

See [DG, Vol. II, Exp. XII, Cor. 1.10] for the following Proposition.

2.3. Proposition. *Let G be a reductive group scheme over a scheme Y. Then the functor on the category of Y-schemes that parametrizes maximal tori of G, is representable by a smooth, separated Y-scheme of finite type. Thus locally in the étale topology of Y, G has split, maximal tori.*

2.3.1. Lemma. *Let Y be an integral scheme. Let G be a reductive group scheme over Y. Let $K := K_Y$. Let $f_K : G_{1K} \to G_K$ be a central isogeny of reductive group schemes over $Spec(K)$. We assume that either G is split or Y is normal. We have:*

(i) *There exists a unique central isogeny $f : G_1 \to G$ that extends $f_K : G_{1K} \to G_K$.*

(b) *If Y is normal, then G_1 is the normalization of G in (the field of fractions of) G_{1K}.*

Proof: We first consider the case when G is split. Let T be a split, maximal torus of G. As f_K is a central isogeny, the inverse image T_{1K} of T_K in G_{1K} is a split torus. Thus G_{1K} is

split. Let $\mathcal{R}_1 \to \mathcal{R}$ be the 1-morphism of root data in the sense of [DG, Vol. III, Exp. XXI, 6.8.1] which is associated to the central isogeny $f_K : G_{1K} \to G_K$ extending the isogeny $T_{1K} \to T_K$. Let $\tilde{f} : \tilde{G}_1 \to G$ be a central isogeny of split, reductive group schemes over Y which extends an isogeny of split tori $\tilde{T}_1 \to T$ and for which the 1-morphism of root data associated to it and to the isogeny $\tilde{T}_1 \to T$, is $\mathcal{R}_1 \to \mathcal{R}$ (cf. [DG, Vol. III, Exp. XXV, Thm. 1.1]). From loc. cit. we also get that there exists an isomorphism $i_{1K} : \tilde{G}_{1K} \tilde{\to} G_{1K}$ such that we have $\tilde{f}_K = f_K \circ i_{1K}$. Obviously, i_{1K} is unique. Let G_1 be the unique group scheme over Y such that i_{1K} extends (uniquely) to an isomorphism $i_1 : \tilde{G}_1 \to G_1$. Let $f := \tilde{f} \circ i_1^{-1}$; it is a central isogeny between G_1 and G. Thus (a) holds if G is split.

Suppose that Y is normal. If G_1 exists, then it is a smooth scheme over the normal scheme Y and therefore it is a normal scheme; from this and the fact that $f : G_1 \to G$ is a finite morphism, we get that G_1 is the normalization of G in G_{1K}.

Thus to end the proof of the Lemma, it suffices to show that the normalization G_1 of G in G_{1K} is a reductive group scheme equipped with a central isogeny $f : G_1 \to G$. This is a local statement for the étale topology of Y. As each connected, étale scheme over Y is a normal, integral scheme, based on Proposition 2.3 we can assume that G has a split, maximal torus T. Thus the fact that G_1 is a reductive group scheme equipped with a central isogeny $f : G_1 \to G$ follows from the previous two paragraphs. $\square$

2.3.2. Lemma. *Let $Y = Spec(A)$ be an affine scheme. Let $K := K_Y$. Let T be a torus over Y equipped with a homomorphism $\rho : T \to GL_M$, where M is a projective A-module which locally has finite rank. We have:*

(a) *the kernel $Ker(\rho)$ is a group scheme over Y of multiplicative type;*

(b) *the kernel $Ker(\rho)$ is trivial (resp. finite) if and only if $Ker(\rho_K)$ is trivial (resp. finite);*

(c) *the quotient group scheme $T/Ker(\rho)$ is a torus and we have a closed embedding homomorphism $T/Ker(\rho) \hookrightarrow GL_M$.*

Proof: The statements of the Lemma are local for the étale topology of Y. Thus we can assume that Y is local and (cf. Proposition 2.3) that T is split. The representation of T on M is a finite direct sum of representations of T of rank 1, cf. [Ja, Part I, Subsection 2.11]. Thus ρ factors as the composite of a homomorphism $\rho_1 : T \to \mathbb{G}_{mA}^m$ with a closed embedding homomorphism $\mathbb{G}_{mA}^m \hookrightarrow GL_M$; here $m \in \mathbb{N}$ is the rank of M. The kernel $\mathrm{Ker}(\rho_1)$ is a group scheme over Y of multiplicative type, cf. [DG, Vol. II, Exp. IX, Prop. 2.7 (i)]. As $\mathrm{Ker}(\rho) = \mathrm{Ker}(\rho_1)$, we get that (a) holds. As (a) holds, $\mathrm{Ker}(\rho)$ is flat over Y as well as the extension of a finite, flat group scheme T_1 by a torus T_0. But T_1 (resp. T_0) is a trivial group scheme if and only if T_{1K} (resp. T_{0K}) is trivial. From this (b) follows. The quotient group scheme $T/\mathrm{Ker}(\rho)$ exists and is a closed subgroup scheme of $\mathbb{G}_{mA}^m$ that is of multiplicative type, cf. [DG, Vol. II, Exp. IX, Prop. 2.7 (i) and Cor. 2.5]. As the fibres of $T/\mathrm{Ker}(\rho)$ are tori, we get that $T/\mathrm{Ker}(\rho)$ is a torus. Thus (c) holds. $\square$

2.4. Lemma. *Suppose that $k = \bar{k}$. Let F be a reductive group over $Spec(k)$. Let $\mathfrak{n}$ be a non-zero ideal of $Lie(F)$ which is a simple left F-module. We assume that there exists a maximal torus T of F such that we have $Lie(T) \cap \mathfrak{n} = 0$. Then $char(k) = 2$ and F^{der} has*

a normal, subgroup F_0 which is isomorphic to SO_{2n+1k} for some $n \in \mathbb{N}^*$ and for which we have an inclusion $\mathfrak{n} \subseteq Lie(F_0)$.

Proof: This is only a variant of [Va5, Lemma 2.1]. $\square$

2.5. Theorem. *Let $f : G_1 \to G_2$ be a homomorphism between group schemes over a scheme Y. We assume that G_1 is reductive, that G_2 is separated and of finite presentation, and that all fibres of f are closed embeddings. Then f is a closed embedding.*

Proof: As G_1 is of finite presentation over Y, the homomorphism f is locally of finite type. As the fibres of f are closed embeddings and thus monomorphisms, f itself is a monomorphism (cf. [DG, Vol. I, Exp. VI_B, Cor. 2.11]). Thus the Theorem follows from [DG, Vol. II, Exp. XVI, Cor. 1.5 a)]. $\square$

2.5.1. Lemma. *Let G be an adjoint group scheme over an affine scheme $Y = Spec(A)$. Let $Aut(G)$ be the group scheme of automorphisms of G. Then the natural adjoint representation $Ad\colon Aut(G) \to GL_{\mathrm{Lie}(G)}$ is a closed embedding.*

Proof: To prove this Lemma, we can work locally in the étale topology of Y and therefore (cf. Proposition 2.3) we can assume G is split and Y is connected. We have a short exact sequence $0 \to G \to \mathrm{Aut}(G) \to C \to 0$ that splits (cf. [DG, Vol. III, Exp. XXIV, Thm. 1.3]), where C is a finite, étale, constant group scheme over Y. Thus G is the identity component of $\mathrm{Aut}(G)$ and $\mathrm{Aut}(G)$ is a finite disjoint union of right translates of G via certain Y-valued points of $\mathrm{Aut}(G)$. If the fibres of Ad are closed embeddings, then the restriction of Ad to G is a closed embedding (cf. Theorem 2.5) and thus also the restriction of Ad to any right translate of G via a Y-valued point of $\mathrm{Aut}(G)$ is a closed embedding. The last two sentences imply that Ad is a closed embedding. Thus to end the proof, we are left to check that the fibres of Ad are closed embeddings. For this, we can assume A is an algebraically closed field.

As G is adjoint and A is a field, the restriction of Ad to G is a closed embedding. Thus the representation Ad is a closed embedding if and only if each element $g \in \mathrm{Aut}(G)(A)$ that acts trivially on $\mathrm{Lie}(G)$, is trivial. We show that the assumption that there exists a non-trivial such element g leads to a contradiction. For this, we can assume that G is absolutely simple and that g is a non-trivial outer automorphism of G. Let T be a maximal torus of a Borel subgroup B of G. Let $t \in \mathrm{Lie}(T)$ be such that its centralizer in G has T as its identity component. As g fixes t and $\mathrm{Lie}(B)$, g normalizes both T and B. But it is well known that a non-trivial outer automorphism g of G that normalizes both T and B, can not fix $\mathrm{Lie}(B)$. Contradiction. Thus Ad is a closed embedding. $\square$

We follow [Va1, Prop. 3.1.2.1 c) and Rm. 3.1.2.2 3)] to prove the next Proposition.

2.5.2. Proposition. *Let V be a discrete valuation ring whose residue field is k. Let $Y = Spec(V)$ and let $K := K_Y$. Let $f : F_1 \to F_2$ be a homomorphism between flat, finite type, affine group schemes over Y such that F_1 is a reductive group scheme and the generic fibre $f_K : F_{1K} \to F_{2K}$ of f is a closed embedding. We have:*

(a) *The subgroup scheme $Ker(f_k : F_{1k} \to F_{2k})$ of F_{1k} has a trivial intersection with each torus T_{1k} of F_{1k}. In particular, we have $Lie(Ker(f_k)) \cap Lie(T_{1k}) = 0$.*

(b) *The homomorphism f is finite.*

(c) *If $char(k) = 2$, we assume that F_{1K} has no normal subgroup that is adjoint of isotypic B_n Dynkin type for some $n \in \mathbb{N}^*$. Then f is a closed embedding.*

Proof: Let $\rho : F_2 \hookrightarrow GL_L$ be a closed embedding homomorphism, with L a free V-module of finite rank (cf. [DG, Vol. I, Exp. VI_B, Rm. 11.11.1]). To prove this Proposition we can assume that V is complete, that $k = \bar{k}$, and that $f_K : F_{1K} \to F_{2K}$ is an isomorphism. Let $F_{0k} := \mathrm{Ker}(f_k)$. We follow [Va1, Rm. 3.1.2.2 3) and proof of Lemma 3.1.6] in order to check that the group scheme $F_{0k} \cap T_{1k}$ is trivial. As V is strictly henselian, the maximal torus T_{1k} of F_{1k} is split and (cf. Proposition 2.3) it lifts to a maximal torus T_1 of F_1. The restriction of $\rho \circ f$ to T_1 has a trivial kernel (as its fibre over K is trivial, cf. Lemma 2.3.2 (b)) and therefore it is a closed embedding (cf. Lemma 2.3.2 (c)). Thus the restriction of f to T_1 is a closed embedding homomorphism $T_1 \hookrightarrow F_2$. Therefore the intersection $F_{0k} \cap T_{1k}$ is a trivial group scheme. Thus (a) holds.

We check (b). The identity component of the reduced scheme of $\mathrm{Ker}(f_k)$ is a reductive group that has 0 rank (cf. (a)) and therefore it is a trivial group. Thus f is a quasi-finite, birational morphism. From Zariski Main Theorem (see [Gr1, Thm. (8.12.6)]) we get that F_1 is an open subscheme of the normalization F_2^{n} of F_2. Let F_3 be the smooth locus of F_2^{n} over $\mathrm{Spec}(V)$; it is an open subscheme of F_2^{n} that contains F_1. It makes sense to speak about the (right or left) translation of F_1 via some V-valued point of F_2^{n}; it is an open subscheme of F_3 and thus we have $F_3(V) = F_2^{\mathrm{n}}(V)$. Moreover, F_3 has a natural structure of a quasi-affine group scheme over $\mathrm{Spec}(V)$ that is of finite type.

As V is complete, it is also excellent (cf. [Ma, §34]). Thus the morphism $F_2^{\mathrm{n}} \to F_2$ is finite. To check that f is finite, it suffices to show that $F_1 = F_2^{\mathrm{n}}$. We show that the assumption that $F_1 \neq F_2^{\mathrm{n}}$ leads to a contradiction. For this we can replace the homomorphism $f : F_1 \to F_2$ of group schemes over $\mathrm{Spec}(V)$ by its pull back to the spectrum of any finite, discrete valuation ring extension of V. Let $x \in F_2^{\mathrm{n}}(k) \setminus F_1(k)$. Based on [Gr1, Cor. (17.16.2)], by using such a replacement of f we can assume that there exists a point $z \in F_2^{\mathrm{n}}(V) = F_3(V)$ which lifts x. We have $z \in F_3(V) \setminus F_1(V)$. From Theorem 2.5 we get that F_1 is a closed subscheme of F_3. Thus, as F_3 is an integral scheme and as $F_{3K} = F_{1K}$, we get that $F_1 = F_3$. This contradicts the fact that $z \in F_3(V) \setminus F_1(V)$. Thus (b) holds.

We check (c). We show that the assumption that $\mathrm{Lie}(F_{0k}) \neq 0$ leads to a contradiction. From Lemma 2.4 applied to F_{1k} and to the left F_{1k}-module $\mathrm{Lie}(F_{0k})$, we get that $\mathrm{char}(k) = 2$ and that F_{1k} has a normal subgroup F_{4k} isomorphic to SO_{2n+1k} for some $n \in \mathbb{N}^*$. As F_{4k} is adjoint, we have a product decomposition $F_{1k} = F_{4k} \times_{\mathrm{Spec}(k)} F_{5k}$ of reductive groups. It lifts (cf. [DG, Vol. III, Exp. XXIV, Prop. 1.21]) to a product decomposition $F_1 = F_4 \times_{\mathrm{Spec}(V)} F_5$, where F_4 is isomorphic to SO_{2n+1V} and where F_5 is a reductive group scheme over $\mathrm{Spec}(V)$. This contradicts the extra hypothesis of (c). Thus we have $\mathrm{Lie}(F_{0k}) = 0$. Therefore F_{0k} is a finite, étale, normal subgroup of F_{1k}. But F_{1k} is connected and thus its action on F_{0k} via inner conjugation is trivial. Therefore we have $F_{0k} \leqslant Z(F_1)_k \leqslant T_{1k}$. Thus $F_{0k} = F_{0k} \cap T_{1k}$ is the trivial group, cf. (a). In other words, the homomorphism $f_k : F_{1k} \to F_{2k}$ is a closed embedding. Thus $f : F_1 \to F$ is a closed embedding homomorphism, cf. Theorem 2.5. $\square$

2.5.3. Remark. See [Va4, Thm. 1.2 (b)] and [PY] for two other proofs of Proposition 2.5.2 (c).

2.6. Definitions. (a) Let $p \in \mathbb{N}^*$ be a prime and let $\mathcal{S}$ be a subset of $\mathbb{Z}$. We say $\mathcal{S}$ is of p-type 1, if the natural map $\mathcal{S} \to \mathbb{Z}/p\mathbb{Z}$ is injective. We say $\mathcal{S}$ is of p-type 2 (resp. of p-type 3), if $\mathcal{S}$ (resp. if $2\mathcal{S}$) is a subset of $\{-p+1, -p+2, \ldots, p-2, p-1\}$.

(b) Let $Y = \mathrm{Spec}(A)$ be a non-empty affine scheme. Let T be a split torus over Y of rank 1. A character of T will be called diagonal with respect to a fixed isomorphism $i_T : T \xrightarrow{\sim} \mathbb{G}_{mY}$, if it is the composite of i_T with the pull back to Y of an endomorphism of $\mathbb{G}_{m\mathbb{Z}}$. We identify the group of diagonal characters of T with respect to i_T with $\mathbb{Z} = \mathrm{End}(\mathbb{G}_{m\mathbb{Z}})$. If Y is connected, then each character of T is a diagonal character with respect to i_T.

2.6.1. On left $\mathfrak{sl}_2$-modules. Let $p \in \mathbb{N}^*$ be a prime. Let A be a commutative $\mathbb{Z}_{(p)}$-algebra. Let M be a projective A-module which locally has finite rank. Let $\mathfrak{g}$ be an $\mathfrak{sl}_2$ Lie algebra over A equipped with a Lie homomorphism $\mathfrak{g} \to \mathfrak{gl}_M$. We consider an A-basis $\{h, x, y\}$ such that the formulas $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ hold. For $s \in \mathbb{N}$, we will view x^s and y^s as A-linear endomorphisms of M (here $x^0 = y^0 := 1_M$).

Let $j \in \{1, \ldots, p-1\}$. Let $z_0 \in M$ be such that we have $h(z_0) = jz_0$ and $x(z_0) = 0$. For $i \in \{1, \ldots, j\}$, let $z_i := \frac{1}{i!} y^i(z_0)$. Then the following formulas hold (see [Hu1, 7.2, Lemma]; the arguments of loc. cit. hold over any base A):

(i) for $i \in \{1, \ldots, j\}$, we have $h(z_i) = (j - 2i)z_i$ and $x(z_i) = (j - i + 1)z_{i-1}$.

From (i) we get that:

(ii) for each $i \in \{0, \ldots, j\}$, the element $x^i(y^i(z_0))$ is a multiple of z_0 by an invertible element of $\mathbb{Z}_{(p)}$ and so also of A.

We have the following general form of [Va1, Claim 3, p. 465].

2.6.2. Proposition. *Let $p \in \mathbb{N}^*$ be a prime. Let $Y = Spec(A)$ be a local $Spec(\mathbb{Z}_{(p)})$-scheme whose residue field k has characteristic p. Let $K := K_Y$. Let M be a free A-module of finite rank. Let T be a split torus over Y that has rank 1 and that is equipped with an isomorphism $i_T : T \xrightarrow{\sim} \mathbb{G}_{mY}$ and with a homomorphism $f_K : T_K \to GL_{M\otimes_A K}$. We assume that we have a direct sum decomposition $M \otimes_A K = \oplus_{i\in\mathcal{S}} M_{iK}$ in free K-modules such that T_K acts on each M_{iK} via a diagonal character i of T_K with respect to $i_T \times_Y Spec(K)$. We also assume that one of the following two conditions hold:*

(i) *the subset $\mathcal{S}$ of $\mathbb{Z}$ is of p-type 1 and the Lie homomorphism $Lie(T_K) \to \mathfrak{gl}_M \otimes_A K$ is the tensorization with K over A of a Lie homomorphism $Lie(T) \to \mathfrak{gl}_M$;*

(ii) *the subset $\mathcal{S}$ of $\mathbb{Z}$ is of p-type 2 and there exists a semisimple group scheme G over Y which is isomorphic to SL_{2Y}, which has T as a maximal torus, and for which there exists a Lie homomorphism $Lie(G) \to \mathfrak{gl}_M$ whose tensorization with K over A is a Lie homomorphism $Lie(G_K) \to \mathfrak{gl}_{M\otimes_A K}$ that is also a morphism of left T_K-modules and that extends the natural Lie homomorphism $Lie(T_K) \to \mathfrak{gl}_{M\otimes_A K}$.*

Then the homomorphism $\rho_K : T_K \to GL_{M\otimes_A K}$ extends to a homomorphism $\rho : T \to GL_M$. If ρ_K is a closed embedding, then ρ is also a closed embedding.

Proof: Let m_A be the maximal ideal of A; we have $k = A/m_A$. Let $M_i := M \cap M_{iK}$, the intersection being taken inside $M \otimes_A K$. We show that the natural injective A-linear map $J : \oplus_{i\in\mathcal{S}} M_i \hookrightarrow M$ is an A-linear isomorphism. It suffices to show that for each $i_0 \in \mathcal{S}$

there exists a projector π_{i_0} of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{iK})$. Let h be the standard generator of $\mathrm{Lie}(T) = \mathrm{Lie}(\mathbb{G}_{mA})$; it acts on M_i as the multiplication with $i \in \mathcal{S} \subseteq \mathbb{Z}$.

We first assume that (i) holds. As $\mathcal{S}$ is of p-type 1, the elements $i \in \mathcal{S}$ are not congruent mod p. Thus for each $i_0 \in \mathcal{S}$ there exists a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ that is a product of linear factors and such that $\pi_{i_0} := f_{i_0}(h)$ is a projector of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{iK})$.

Suppose that (ii) holds. To ease the notations, for $i \in \mathbb{Z} \setminus \mathcal{S}$ let $M_i := 0$ and let $\pi_i : M \to M$ be the zero map. Thus we can assume that $\mathcal{S} = \{-p+1, \ldots, p-1\}$. As G is isomorphic to SL_{2Y} and as T is a maximal torus of G, there exists an A-basis $\{h, x, y\}$ for $\mathrm{Lie}(G)$ which contains the standard generator h of $\mathrm{Lie}(T)$ and for which the formulas $[h, x] = 2x$, $[h, y] = -2y$, and $[x, y] = h$ hold. Moreover, M is a left $\mathrm{Lie}(G)$-module. As the Lie homomorphism $\mathrm{Lie}(G_K) \to \mathfrak{gl}_{M \otimes_A K}$ is also a homomorphism of left T_K-modules, we can assume that x and y are such that for $i \in \mathcal{S}$ we have $x(M_{iK}) \subseteq M_{i+2K}$ and $y(M_{iK}) \subseteq M_{i-2K}$ (for $p > 2$ these inclusions hold automatically). Thus for $i \in \mathcal{S}$ we have $x(M_i) \subseteq M_{i+2}$ and $y(M_i) \subseteq M_{i-2}$.

We will show the existence of the projectors π_{i_0} by induction on the rank M. Let $r \in \mathbb{N}^*$. If $r = 1$ and the rank of M is at most r, then the existence of the projectors π_{i_0} is obvious. Suppose that the projectors π_{i_0} exist if the rank of M is less than r. We show that the projectors π_{i_0} exist even if the rank of M is r. For $i \in \{1, \ldots, p-1\}$, we define $N_i := M \cap (M_{iK} + M_{-p+iK})$ and we consider the following two statements:

$Q(i)$ if $(z_i, z_{-p+i}) \in (M_i \setminus m_A M_i) \times M_{-p+i}$, then $z_i + z_{-p+i} \notin m_A N_i$;

$Q(-p+i)$ if $(z_i, z_{-p+i}) \in M_i \times (M_{-p+i} \setminus m_A M_{-p+i})$, then $z_i + z_{-p+i} \notin m_A N_i$.

Let $i_0 \in \{0, \ldots, p-1\}$. If $i_0 = 0$, then as above we argue that there exists a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ that is a product of linear factors and such that $\pi_{i_0} := f_{i_0}(h)$ is a projector of M on M_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0\}} M_{iK})$; thus π_{i_0} exists if $i_0 = 0$. Suppose that $i_0 > 0$. We consider a polynomial $f_{i_0}(x) \in \mathbb{Z}_{(p)}[x]$ such that $f_{i_0}(h)$ is an A-linear endomorphism of M whose kernel is $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0, -p+i_0\}} M_{iK})$ and whose restriction to N_{i_0} is a scalar automorphism of N_{i_0}. We have $\mathrm{Im}(f_{i_0}(h)) = N_{i_0}$ and thus there exists a projector $\pi_{i_0, -p+i_0}$ of M on N_{i_0} along $M \cap (\oplus_{i \in \mathcal{S} \setminus \{i_0, -p+i_0\}} M_{iK})$. Thus N_{i_0} is a direct summand of M and therefore (as Y is local) it is a free A-module of finite rank. Thus π_{i_0} and π_{-p+i_0} exists if and only if the natural injective A-linear map $J_{i_0} : M_{i_0} \oplus M_{-p+i_0} \to N_{i_0}$ is an A-linear isomorphism (i.e., it is onto). But J_{i_0} is onto if and only if $J_{i_0} \otimes 1_k$ is onto. As $J_{i_0} \otimes 1_K$ is a K-linear isomorphism, the free A-modules $M_{i_0} \oplus M_{-p+i_0}$ and N_{i_0} have equal ranks. Thus by reasons of dimensions, we get that the k-linear map $J_{i_0} \otimes 1_k$ is onto if and only if it is injective. Thus, as A is a local ring, the existence of the projectors π_{i_0} and π_{-p+i_0} is equivalent to the fact that both statements $Q(i_0)$ and $Q(-p+i_0)$ hold.

We will check that the statement $Q(i_0)$ holds by decreasing induction on $i_0 \in \{1, \ldots, p-1\}$. Thus we can assume that the statement $Q(i_1)$ holds for all numbers $i_1 \in \{i_0+1, \ldots, p-1\}$. As the statement $Q(i_1)$ holds, the k-linear map $M_{i_1}/m_A M_{i_1} \to M/m_A M$ is injective. To check that the statement $Q(i_0)$ holds, we can replace M by $M/\tilde{M}$, where $\tilde{M}$ is an arbitrary direct summand of M that has trivial intersection with M_{-p+i_0}, that satisfies the identity $\tilde{M} = \oplus_{i \in \mathcal{S}} \tilde{M} \cap M_{iK}$, and that is a left $\mathrm{Lie}(G)$-module. Let $j_0 \in \mathbb{N}$ be the greatest number such that $i_0 + 2j_0 \in \mathcal{S}$ and $M_{i_0+2j_0} \neq 0$. Depending on the value of j_0, we distinguish two cases.

Case 1: $j_0 > 0$**.** As $i_0 + 2j_0 > i_0 > 0$ and $M_{i_0+2j_0} \neq 0$, we have $p > 2$ and the k-linear map $M_{i_0+2j_0}/m_A M_{i_0+2j_0} \to M/m_A M$ is injective and non-zero. Therefore there exists a non-zero element $z_0 \in M_{i_0+2j_0}$ such that $\tilde{M}_{i_0+2j_0} := Az_0$ is a direct summand of both $M_{i_0+2j_0}$ and M. We have $x(\tilde{M}_{i_0+2j_0}) = 0$ (as otherwise $M_{i_0+2j_0+2} \neq 0$). Let $\tilde{M} := \oplus_{s=0}^{i_0+2j_0} y^s(\tilde{M}_{i_0+2j_0}) = \oplus_{s=0}^{i_0+2j_0} \frac{1}{s!} y^s(\tilde{M}_{i_0+2j_0})$. From properties 2.6.1 (i) and (ii) we easily get that for $1 \leq s \leq i_0 + 2j_0$ we have $x(y^s(\tilde{M}_{i_0+2j_0})) = y^{s-1}(\tilde{M}_{i_0+2j_0})$. Moreover we have $y^s(\tilde{M}_{i_0+2j_0}) \subseteq M_{i_0+2j_0-2s}$. These properties of $y^s(\tilde{M}_{i_0+2j_0})$, $\tilde{M}$, and z_0 imply that $x(\tilde{M}) \subseteq \tilde{M}$, that $h(\tilde{M}) \subseteq \tilde{M}$, and that each $y^s(\tilde{M}_{i_0+2j_0})$ with $0 \leq s \leq i_0 + 2j_0$ is a direct summand of M and thus also of $M_{i_0+2j_0-2s}$. As $p > 2$, the numbers $i_0 + 2j_0 - 2s$ with $0 \leq s \leq i_0 + 2j_0$ are not congruent mod p. We easily get that $\tilde{M}$ is a direct summand of M. From property 2.6.1 (i) we get that $\tilde{M} \cap M_{-p+i_0} = 0$.

To check that $\tilde{M}$ is a left Lie(G)-module, it suffices to show that $y^{i_0+2j_0+1}(z_0) = 0$. To check this, we can assume that $M = \oplus_{s=0}^{[\frac{i_0+2j_0+p}{2}]} y^s(\tilde{M}_{i_0+2j_0})$. We show that the assumption that $y^{i_0+2j_0+1}(z_0) \neq 0$ leads to a contradiction. Let $s_0 \in \mathbb{N}^*$ be the greatest number such that $y^{i_0+2j_0+s_0}(z_0) \neq 0$. We have $-i_0 - 2j_0 - 2s_0 \in \mathbb{S}$ and thus $-p+1 \leq -i_0 - 2j_0 - 2s_0 \leq -1$. By applying property 2.6.1 (ii) to the quintuple $(y^{i_0+2j_0+s_0}(z_0), -h, y, x, i_0 + 2j_0 + 2s_0)$ instead of to the quintuple (z_0, h, x, y, j), we get that $w_0 := x^{i_0+2j_0+2s_0}(y^{i_0+2j_0+s_0}(z_0))$ is a non-zero element of M. Thus the element $u_0 := x^{i_0+2j_0+s_0}(y^{i_0+2j_0+s_0}(z_0))$ is a multiple of z_0 by a non-zero element of A. As $w_0 = x^{s_0}(u_0) \neq 0$ and as $s_0 \geq 1$, we have $x(u_0) \neq 0$. Thus there exists a multiple of $x(z_0)$ by an element of A which is non-zero. This contradicts the fact that $x(z_0) = 0$. Thus $\tilde{M}$ is a left Lie(G)-module.

We know that to check that the statement $Q(i)$ holds, we can replace M by $M/\tilde{M}$. But the rank of $M/\tilde{M}$ is less than r and thus by our induction on ranks we know that the analogues of the projectors π_{i_0} exist. Therefore the analogues of the statements $Q(i)$ with $i \in \{1, \ldots, p-1\}$ hold in the context of the left Lie(G)-module $M/\tilde{M}$. Based on the last three sentences, we conclude that the statement $Q(i_0)$ holds if $j_0 > 0$.

Case 2: $j_0 = 0$**.** Suppose that there exist elements $z_{i_0} \in M_{i_0} \setminus m_A M_{i_0}$ and $z_{-p+i_0} \in M_{-p+i_0}$ such that we have $z_{i_0} + z_{-p+i_0} \in m_A N_{i_0}$. We have $y^{i_0}(m_A N_{i_0}) \subseteq m_A M_{-i_0}$ and $y^{i_0}(z_{-p+i_0}) = 0$. Thus $x^{i_0}(y^{i_0}(z_{i_0})) = x^{i_0}(y^{i_0}(z_{i_0} + z_{-p+i_0})) \in x^{i_0}(m_A M_{-i_0}) \subseteq m_A M_{i_0}$. But from property 2.6.1 (ii) we get that $x^{i_0}(y^{i_0}(z_{i_0}))$ is a multiple of z_{i_0} by an invertible element of A. Thus $z_{i_0} \in m_A M_{i_0}$. Contradiction. Thus the statement $Q(i_0)$ holds if $j_0 = 0$.

This ends our decreasing induction on $i_0 \in \{1, \ldots, p-1\}$. As statement $Q(i)$ holds for all $i \in \{1, \ldots, p-1\}$, the statement $Q(-p+i)$ also holds for all $i \in \{1, \ldots, p-1\}$. This is so as Lie(G) has an automorphism that takes the triple (h, x, y) into the triple $(-h, y, x)$ (under this isomorphism the statement $Q(-p+i)$ gets replaced by the statement $Q(p-i)$). This implies that the projectors π_i and π_{-p+i} exists for all $i \in \{1, \ldots, p-1\}$. Therefore all the projectors π_i with $i \in \mathbb{S}$ exists.

Thus, regardless of which one of the two properties (i) and (ii) holds, the A-linear map $J : \oplus_{i \in \mathbb{S}} M_i \hookrightarrow M$ is an A-linear isomorphism. As Y is local, each M_i with $i \in \mathbb{S}$ is a free A-module. Thus we have a unique homomorphism $\rho : T \to GL_M$ such that T acts on the direct summand M_i of M via the diagonal character i of T with respect to i_T. The last part of the Proposition is implied by Lemma 2.3.2 (b) and (c). $\square$

3. Lie algebras with perfect Killing forms

Let A be a commutative $\mathbb{Z}$-algebra. Let $\mathfrak{g}$ be a Lie algebra over A which as an A-module is projective and locally has finite rank. In this Section we will assume that the Killing form $\mathcal{K}_{\mathfrak{g}}$ on $\mathfrak{g}$ is perfect. Let $U_{\mathfrak{g}}$ be the enveloping algebra of $\mathfrak{g}$ i.e., the quotient of the tensor algebra $T_{\mathfrak{g}}$ of $\mathfrak{g}$ by the two-sided ideal of $T_{\mathfrak{g}}$ generated by the subset $\{x\otimes y - y\otimes x - [x,y] | x,y \in \mathfrak{g}\}$ of $T_{\mathfrak{g}}$. Let $Z(U_{\mathfrak{g}})$ be the center of $U_{\mathfrak{g}}$. The categories of left $\mathfrak{g}$-modules and of left $U_{\mathfrak{g}}$-modules are canonically identified. We view $\mathfrak{g}$ as a left $\mathfrak{g}$-module via the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$; let $\mathrm{ad} : U_{\mathfrak{g}} \to \mathrm{End}(\mathfrak{g})$ be the A-homomorphism corresponding to the left $\mathfrak{g}$-module $\mathfrak{g}$. We refer to [CE, Ch. XIII] for the cohomology groups $H^i(\mathfrak{g},\mathfrak{m})$ of a left $\mathfrak{g}$-module $\mathfrak{m}$. We denote also by $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g}\otimes_A \mathfrak{g} \to A$ the A-linear map defined by $\mathcal{K}_{\mathfrak{g}} : \mathfrak{g}\times\mathfrak{g} \to A$. Thus we have $\mathcal{K}_{\mathfrak{g}} \in (\mathfrak{g}\otimes_A \mathfrak{g})^* = \mathfrak{g}^*\otimes_A \mathfrak{g}^*$. Let $\phi : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$ be the A-linear isomorphism defined naturally by $\mathcal{K}_{\mathfrak{g}}$. It induces an A-linear isomorphism $\phi^{-1}\otimes\phi^{-1} : \mathfrak{g}^*\otimes_A \mathfrak{g}^* \xrightarrow{\sim} \mathfrak{g}\otimes_A \mathfrak{g}$. The image Ω of $\phi^{-1}\otimes\phi^{-1}(\mathcal{K}_{\mathfrak{g}}) \in \mathfrak{g}\otimes_A\mathfrak{g} \subseteq T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$ is called the Casimir element of the adjoint representation $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathfrak{g}}$.

3.1. Lemma. *The following four properties hold:*

(a) *if the A-module $\mathfrak{g}$ is free and if $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_m\}$ are two A-bases for $\mathfrak{g}$ such that for all $i,j\in\{1,\ldots,m\}$ we have $\mathcal{K}_{\mathfrak{g}}(x_i\otimes y_j)=\delta_{ij}$, then Ω is the image of the element $\sum_{i=1}^m x_i\otimes y_i$ of $T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$;*

(b) *we have $\Omega\in Z(U_{\mathfrak{g}})$;*

(c) *the Casimir element is fixed by the group of Lie automorphisms of $\mathfrak{g}$ (i.e., if $\sigma : U_{\mathfrak{g}} \xrightarrow{\sim} U_{\mathfrak{g}}$ is the A-algebra automorphism induced by a Lie algebra automorphism $\sigma : \mathfrak{g}\xrightarrow{\sim}\mathfrak{g}$, then we have $\sigma(\Omega)=\Omega$);*

(d) *the Casimir element Ω acts identically on $\mathfrak{g}$ (i.e., ad$(\Omega)=1_{\mathfrak{g}}$).*

Proof: Parts (a) and (b) are well known facts on tensor and enveloping algebras; for instance, if A is reduced, then to check (a) and (b) we can assume that A is a field and for this case we refer to [Bou1, Ch. I, §2, 7, Prop. 11]. To check (c) and (d), we can assume that the A-module $\mathfrak{g}$ is free. Let $\{x_1,\ldots,x_m\}$ and $\{y_1,\ldots,y_m\}$ be two A-bases for $\mathfrak{g}$ as in (a). Thus Ω is the image of $\sum_{i=1}^m x_i\otimes y_i$ in $U_{\mathfrak{g}}$. Therefore $\sigma(\Omega)$ is the image of $\sum_{i=1}^m \sigma(x_i)\otimes\sigma(y_i)$ in $U_{\mathfrak{g}}$. As for $i,j\in\{1,\ldots,m\}$ we have $\mathcal{K}_{\mathfrak{g}}(\sigma(x_i),\sigma(y_j))=\delta_{ij}$, from (a) we get that the image of $\sum_{i=1}^m \sigma(x_i)\otimes\sigma(y_i)\in T_{\mathfrak{g}}$ in $U_{\mathfrak{g}}$ is Ω. Thus $\sigma(\Omega)=\Omega$.

We check (d). Let $z,w\in\mathfrak{g}$. We write $\mathrm{ad}(z)\circ\mathrm{ad}(w)(x_i)=\sum_{j=1}^m a_{ji}x_j$, with a_{ji}'s in A. We recall that $\mathcal{K}_{\mathfrak{g}}$ is $\mathfrak{g}$-invariant i.e., for all $a,b,c\in\mathfrak{g}$ we have an identity $\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(a)(b),c)+\mathcal{K}_{\mathfrak{g}}(b,\mathrm{ad}(a)(c))=0$ (see [Bou1, Ch. I, §6, (13) and Prop. 8]). Using this we compute that:

$$\mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(\Omega)(z),w)=\mathcal{K}_{\mathfrak{g}}(\sum_{i=1}^m \mathrm{ad}(x_i)\circ\mathrm{ad}(y_i)(z),w)=-\sum_{i=1}^m \mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(y_i)(z),\mathrm{ad}(x_i)(w))$$

$$=\sum_{i=1}^m \mathcal{K}_{\mathfrak{g}}(\mathrm{ad}(z)(y_i),\mathrm{ad}(x_i)(w))=-\sum_{i=1}^m \mathcal{K}_{\mathfrak{g}}(y_i,\mathrm{ad}(z)\circ\mathrm{ad}(x_i)(w))$$

$$= \sum_{i=1}^{m} \mathcal{K}_{\mathfrak{g}}(y_i, \mathrm{ad}(z) \circ \mathrm{ad}(w)(x_i)) = \sum_{i,j=1}^{m} a_{ji}\delta_{ji} = \sum_{i=1}^{m} a_{ii} = \mathcal{K}_{\mathfrak{g}}(z,w)$$

(the last equality due to the very definition of $\mathcal{K}_{\mathfrak{g}}$). This implies that for each $z \in \mathfrak{g}$, we have $\mathrm{ad}(\Omega)(z) - z \in \mathrm{Ker}(\mathcal{K}_{\mathfrak{g}}) = 0$. Thus $\mathrm{ad}(\Omega)(z) = z$ i.e., (d) holds. $\square$

3.2. Fact. *Let $i \in \mathbb{N}$. Let $\mathfrak{m}$ be a left $\mathfrak{g}$-module on which Ω acts identically. Then the cohomology group $H^i(\mathfrak{g}, \mathfrak{m})$ is trivial.*

Proof: We have an identity $H^i(\mathfrak{g}, \mathfrak{m}) = \mathrm{Ext}^i_{U_{\mathfrak{g}}}(A, \mathfrak{m})$ of $Z(U_{\mathfrak{g}})$-modules, cf. [CE, Ch. XIII, §2 and §8]. As $\Omega \in Z(U_{\mathfrak{g}})$ acts trivially on A and identically on $\mathfrak{m}$, the group $\Omega\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A, \mathfrak{m})$ is on one hand trivial and on the other hand it is $\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A, \mathfrak{m})$. Thus $\mathrm{Ext}^i_{U_{\mathfrak{g}}}(A, \mathfrak{m}) = 0$. Therefore $H^i(\mathfrak{g}, \mathfrak{m}) = 0$. $\square$

3.3. Theorem. *The group scheme $Aut(\mathfrak{g})$ over $Spec(A)$ of Lie automorphisms of $\mathfrak{g}$ is smooth and locally of finite presentation.*

Proof: To check this, we can assume that the A-module $\mathfrak{g}$ is free. The group scheme $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup scheme of $GL_{\mathfrak{g}}$ defined by a finitely generated ideal of the ring of functions of $GL_{\mathfrak{g}}$. Thus $\mathrm{Aut}(\mathfrak{g})$ is of finite presentation. Thus to show that $\mathrm{Aut}(\mathfrak{g})$ is smooth over $\mathrm{Spec}(A)$, it suffices to show that for each affine morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ and for each ideal J of B such that $J^2 = 0$, the restriction map $\mathrm{Aut}(\mathfrak{g})(B) \to \mathrm{Aut}(\mathfrak{g})(B/J)$ is onto (cf. [BLR, Ch. 2, 2.2, Prop. 6]). Not to introduce extra notations by repeatedly tensoring with B over A, we will assume that $B = A$. Thus J is an ideal of A and we have to show that the restriction map $\mathrm{Aut}(\mathfrak{g})(A) \to \mathrm{Aut}(\mathfrak{g})(A/J)$ is onto.

Let $\bar{\sigma} : \mathfrak{g}/J\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}/J\mathfrak{g}$ be a Lie automorphism. Let $\sigma_0 : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be an A-linear automorphism that lifts $\bar{\sigma}$. Let $J\mathfrak{g}_{\bar{\sigma}}$ be the left $\mathfrak{g}$-module which as an A-module is $J\mathfrak{g}$ and whose left $\mathfrak{g}$-module structure is defined as follows: if $x \in \mathfrak{g}$, then x acts on $J\mathfrak{g}_{\bar{\sigma}}$ in the same way as $\mathrm{ad}(\bar{\sigma}(x))$ (equivalently, as $\mathrm{ad}(\sigma_0(x))$) acts on the A-module $J\mathfrak{g} = J\mathfrak{g}_{\bar{\sigma}}$; this makes sense as $J^2 = 0$. Let $\theta : \mathfrak{g} \times \mathfrak{g} \to J\mathfrak{g}_{\bar{\sigma}}$ be the alternating map defined by the rule:

$$\theta(x,y) := [\sigma_0(x), \sigma_0(y)] - \sigma_0([x,y]) \quad \forall\, x, y \in \mathfrak{g}. \tag{1}$$

We check that θ is a 2-cocycle i.e., for all $x, y, z \in \mathfrak{g}$ we have an identity

$$d\theta(x,y,z) := x(\theta(y,z)) - y(\theta(x,z)) + z(\theta(x,y)) - \theta([x,y],z) + \theta([x,z],y) - \theta([y,z],x) = 0.$$

The expression $d\theta$ is a sum of 12 terms which can be divided into three groups. The first group contains the three terms $-\sigma_0([[x,y],z])$, $\sigma_0([[x,z],y])$, and $-\sigma_0([[y,z],x])$; their sum is 0 due to the Jacobi identity and the fact that σ_0 is an A-linear map. The second group contains the six terms $[\sigma_0(x), \sigma_0[y,z]]$, $-[\sigma_0(x), \sigma_0[y,z]]$, $[\sigma_0(y), \sigma_0[x,z]]$, $-[\sigma_0(y), \sigma_0[x,z]]$, $[\sigma_0(z), \sigma_0[x,y]]$, $-[\sigma_0(z), \sigma_0[x,y]]$; obviously their sum is 0. The third group contains the three terms $[\sigma_0(x), [\sigma_0(y), \sigma_0(z)]]$, $-[\sigma_0(y), [\sigma_0(x), \sigma_0(z)]]$, and $[\sigma_0(z), [\sigma_0(x), \sigma_0(y)]]$; their sum is 0 due to the Jacobi identity. Thus indeed $d\theta = 0$.

As Ω (i.e., $\mathrm{ad}(\Omega)$) acts identically on $\mathfrak{g}$ (cf. Lemma 3.1 (d)), it also acts identically on $J\mathfrak{g}$. But Ω mod J is fixed by the Lie automorphism $\bar{\sigma}$ of $\mathfrak{g}/J\mathfrak{g}$, cf. Lemma 3.1 (c). Thus

Ω also acts identically on the left $\mathfrak{g}$-module $J\mathfrak{g}_{\bar\sigma}$. From this and the Fact 3.2 we get that $H^2(\mathfrak{g}, J\mathfrak{g}_{\bar\sigma}) = 0$. Thus θ is the coboundary of a 1-cochain $\delta : \mathfrak{g} \to J\mathfrak{g}_{\bar\sigma}$ i.e., we have

$$\theta(x,y) = x(\delta(y)) - y(\delta(x)) - \delta([x,y]) \quad \forall x,\, y \in \mathfrak{g}. \tag{2}$$

Let $\sigma : \mathfrak{g} \xrightarrow{\sim} \mathfrak{g}$ be the A-linear isomorphism defined by the rule $\sigma(x) := \sigma_0(x) - \delta(x)$; here $\delta(x)$ is an element of the A-module $J\mathfrak{g} = J\mathfrak{g}_{\bar\sigma}$. Due to formulas (1) and (2), we compute

$$\sigma([x,y]) = \sigma_0([x,y]) - \delta([x,y]) = [\sigma_0(x), \sigma_0(y)] - \theta(x,y) - \delta([x,y])$$

$$= [\sigma_0(x), \sigma_0(y)] - x(\delta(y)) + y(\delta(x)) = [\sigma_0(x), \sigma_0(y)] - \mathrm{ad}(\bar\sigma(x))(\delta(y)) + \mathrm{ad}(\bar\sigma(y))(\delta(x))$$

$$= [\sigma_0(x), \sigma_0(y)] - \mathrm{ad}(\sigma_0(x))(\delta(y)) + \mathrm{ad}(\sigma_0)(y)(\delta(x)) = [\sigma_0(x), \sigma_0(y)] - [\sigma_0(x), \delta(y)] + [\sigma_0(y), \delta(x)]$$

$$= [\sigma_0(x) - \delta(x), \sigma_0(y) - \delta(y)] - [\delta(x), \delta(y)] = [\sigma(x), \sigma(y)] - [\delta(x), \delta(y)] = [\sigma(x), \sigma(y)]$$

(the last identity as $J^2 = 0$). Thus σ is a Lie automorphism of $\mathfrak{g}$ that lifts the Lie automorphism $\bar\sigma$ of $\mathfrak{g}/J\mathfrak{g}$. Thus the restriction map $\mathrm{Aut}(\mathfrak{g})(A) \to \mathrm{Aut}(\mathfrak{g})(A/J)$ is onto. $\square$

3.4. Two groupoids on sets. Let Y be a scheme. Let Adj-perf_Y be the category whose:

– objects are adjoint group schemes over Y with the property that their Lie algebras $\mathcal{O}_Y$-modules have perfect Killing forms;

– morphisms are isomorphisms of group schemes.

Let Lie-perf_Y be the category whose:

– objects are Lie algebras $\mathcal{O}_Y$-modules which as $\mathcal{O}_Y$-modules are locally free of finite rank and whose Killing forms are perfect;

– morphisms are isomorphisms of Lie algebras.

We recall that, as all morphisms of Adj-perf_Y and Lie-perf_Y are isomorphisms, the categories Adj-perf_Y and Lie-perf_Y are (called) groupoids on sets.

3.5. Theorem. *Let $\mathcal{L} : \text{Adj-perf}_Y \to \text{Lie-perf}_Y$ be the functor which associates to a morphism $f : G \xrightarrow{\sim} H$ of Adj-perf_Y the morphism $df : \mathrm{Lie}(G) \xrightarrow{\sim} \mathrm{Lie}(H)$ of Lie-perf_Y which is the differential of f. Then the functor $\mathcal{L}$ is an equivalence of categories.*

Proof: The functor $\mathcal{L}$ is faithful, cf. Lemma 2.5.1. Thus to prove the Theorem it suffices to show that $\mathcal{L}$ is surjective on objects and that $\mathcal{L}$ is fully faithful. To check this, as Adj-perf_Y and Lie-perf_Y are groupoids on sets and as $\mathcal{L}$ is faithful, we can assume that $Y = \mathrm{Spec}(A)$ is affine. Thus to end the proof of the Theorem it suffices to prove the following two things:

(i) there exists a unique open subgroup scheme $\mathrm{Aut}(\mathfrak{g})^0$ of $\mathrm{Aut}(\mathfrak{g})$ which is an adjoint group scheme over Y and whose Lie algebra is the Lie subalgebra $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{gl}_{\mathfrak{g}}$ (therefore $\mathfrak{g} = \mathrm{ad}(\mathfrak{g})$ is the image through $\mathcal{L}$ of the object $\mathrm{Aut}(\mathfrak{g})^0$ of Adj-perf_Y);

(ii) the group scheme $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ of automorphisms of $\mathrm{Aut}(\mathfrak{g})^0$ is $\mathrm{Aut}(\mathfrak{g})$ acting on $\mathrm{Aut}(\mathfrak{g})^0$ via inner conjugation (therefore $\mathrm{Aut}(\mathfrak{g})(A) = \mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)(A)$).

To check these two properties, we can assume that the A-module $\mathfrak{g}$ is free of rank $m \in \mathbb{N}^*$. Let k be the residue field of an arbitrary point y of Y. The Lie algebra $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k)$ is the Lie algebra of derivations of $\mathfrak{g} \otimes_A k$. As the Killing form $\mathcal{K}_{\mathfrak{g}\otimes_A k}$ is perfect, as in [Hu1, 5.3, Thm.] one argues that each derivation of $\mathfrak{g} \otimes_A k$ is an inner derivation. Thus we have $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$. As the group scheme $\mathrm{Aut}(\mathfrak{g})$ over Y is smooth and locally of finite presentation (cf. Theorem 3.3), from [DG, Vol. I, Exp. VI_B, Cor. 4.4] we get that there exists a unique open subgroup scheme $\mathrm{Aut}(\mathfrak{g})^0$ of $\mathrm{Aut}(\mathfrak{g})$ whose fibres are connected. The fibres of $\mathrm{Aut}(\mathfrak{g})^0$ are open-closed subgroups of the fibres of $\mathrm{Aut}(\mathfrak{g})$ and therefore are affine.

Let N_k be a smooth, connected, unipotent, normal subgroup of $\mathrm{Aut}(\mathfrak{g})^0_k$. The Lie algebra $\mathrm{Lie}(N_k)$ is a nilpotent ideal of $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$. Thus $\mathrm{Lie}(N_k) \subseteq \mathrm{Ker}(\mathcal{K}_{\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k)}) = \mathrm{Ker}(\mathcal{K}_{\mathrm{ad}(\mathfrak{g})\otimes_A k})$, cf. [Bou1, Ch. I, §4, Prop. 6 (b)]. As the Killing form $\mathcal{K}_{\mathrm{ad}(\mathfrak{g})\otimes_A k}$ is perfect, we have $\mathrm{Lie}(N_k) = 0$. Thus N_k is the trivial subgroup of $\mathrm{Aut}(\mathfrak{g})^0_k$ and therefore the unipotent radical of $\mathrm{Aut}(\mathfrak{g})^0_k$ is trivial. Thus $\mathrm{Aut}(\mathfrak{g})^0_k$ is an affine, connected, smooth group over k whose unipotent radical is trivial. Therefore $\mathrm{Aut}(\mathfrak{g})^0_k$ is a reductive group over $\mathrm{Spec}(k)$, cf. [Bo, Subsection 11.21]. As $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$ has trivial center, the group $\mathrm{Aut}(\mathfrak{g})^0_k$ is semisimple. Thus the smooth group scheme $\mathrm{Aut}(\mathfrak{g})^0$ of finite presentation over Y has semisimple fibres. Thus $\mathrm{Aut}(\mathfrak{g})^0$ is a semisimple group scheme over Y, cf. [DG, Vol. II, Exp. XVI, Thm. 5.2 (ii)]. As $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ acts trivially on $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k$ and as $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ is a subgroup of $\mathrm{Aut}(\mathfrak{g})_k$, the group $Z(\mathrm{Aut}(\mathfrak{g})^0)_k$ is trivial. This implies that finite, flat group scheme $Z(\mathrm{Aut}(\mathfrak{g})^0)$ is trivial and therefore $\mathrm{Aut}(\mathfrak{g})^0$ is an adjoint group scheme.

The Lie subalgebras $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)$ and $\mathrm{ad}(\mathfrak{g})$ of $\mathfrak{gl}_{\mathfrak{g}}$ are free A-submodules of the Lie subalgebra $\mathfrak{l}$ of $\mathfrak{gl}_{\mathfrak{g}}$ formed by derivations of $\mathfrak{g}$. As for each point y of Y we have $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_k) = \mathrm{ad}(\mathfrak{g}) \otimes_A k = \mathfrak{l} \otimes_A k$, $\mathfrak{l}$ is locally generated by either $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)$ or $\mathrm{ad}(\mathfrak{g})$. We easily get that we have identities $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{ad}(\mathfrak{g}) = \mathfrak{l}$.

The group scheme $\mathrm{Aut}(\mathfrak{g})$ acts via inner conjugation on $\mathrm{Aut}(\mathfrak{g})^0$. As $\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{ad}(\mathfrak{g})$ and as $\mathrm{Aut}(\mathfrak{g})$ is a closed subgroup scheme of $GL_{\mathfrak{g}}$, the inner conjugation homomorphism $\mathrm{Aut}(\mathfrak{g}) \to \mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ has trivial kernel. As $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ is a closed subgroup scheme of $\mathrm{Aut}(\mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)) = \mathrm{Aut}(\mathrm{ad}(\mathfrak{g}))$ (cf. Lemma 2.5.1), we can identify naturally $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0)$ with a closed subgroup scheme of $\mathrm{Aut}(\mathfrak{g})$. From the last two sentences, we get that $\mathrm{Aut}(\mathrm{Aut}(\mathfrak{g})^0) = \mathrm{Aut}(\mathfrak{g})$. Thus both sentences (i) and (ii) hold. $\square$

The next Proposition details on the range of applicability of the Theorem 3.5.

3.6. Proposition. **(a)** *We recall k is a field. Let F be a semisimple group over Spec(k). Then the Killing form $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect if and only if the following two conditions hold:*

(i) *either char$(k) = 0$ or char(k) is an odd prime p and F^{ad} has no simple factor of isotypic A_{pn-1}, $B_{pn+\frac{1-p}{2}}$, C_{pn-1}, or D_{pn+1} Dynkin type (here $n \in \mathbb{N}^*$);*

(ii) *if char$(k) = 3$ (resp. if char$(k) = 5$), then F^{ad} has no simple factor of isotypic E_6, E_7, E_8, F_4, G_2 (resp. of isotypic E_8) Dynkin type.*

(b) *If $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect, then the central isogenies $F^{\mathrm{sc}} \to F \to F^{\mathrm{ad}}$ are étale; thus, by identifying tangent spaces at identity elements, we have Lie(F^{sc}) = Lie(F) = Lie(F^{ad}).*

Proof: We can assume that $k = \bar{k}$ and that $\mathrm{tr.deg.}(k) < \infty$. If $\mathrm{char}(k) = 0$, then $\mathrm{Lie}(F)$ is a semisimple Lie algebra over k and therefore the Proposition follows from [Hu1, 5.1, Thm.]. Thus we can assume $\mathrm{char}(k)$ is a prime $p \in \mathbb{N}^*$. If (i) and (ii) hold, then p does not divide the order of the finite group scheme $Z(F^{\mathrm{sc}})$ (see [Bou2, plates I to IX]) and therefore (a) implies (b).

Let $W(k)$ be the ring of Witt vectors with coefficients in k. Let $F_{W(k)}$ be a semisimple group scheme over $\mathrm{Spec}(W(k))$ that lifts F, cf. [DG, Vol. III, Exp. XXIV, Prop. 1.21]. We have identities $\mathrm{Lie}(F^{\mathrm{sc}}_{W(k)})[\frac{1}{p}] = \mathrm{Lie}(F_{W(k)})[\frac{1}{p}] = \mathrm{Lie}(F^{\mathrm{ad}}_{W(k)})[\frac{1}{p}]$. This implies that:

(iii) $\mathcal{K}_{\mathrm{Lie}(F_{W(k)})}$ is the composite of the natural $W(k)$-linear map $\mathrm{Lie}(F_{W(k)}) \times \mathrm{Lie}(F_{W(k)}) \to \mathrm{Lie}(F^{\mathrm{ad}}_{W(k)}) \times \mathrm{Lie}(F^{\mathrm{ad}}_{W(k)})$ with $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{ad}}_{W(k)})}$;

(iv) $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}}_{W(k)})}$ is the composite of the natural $W(k)$-linear map $\mathrm{Lie}(F^{\mathrm{sc}}_{W(k)}) \times \mathrm{Lie}(F^{\mathrm{sc}}_{W(k)}) \to \mathrm{Lie}(F_{W(k)}) \times \mathrm{Lie}(F_{W(k)})$ with $\mathcal{K}_{\mathrm{Lie}(F_{W(k)})}$.

We prove (a). We have $\mathrm{Ker}(\mathrm{Lie}(F) \to \mathrm{Lie}(F^{\mathrm{ad}})) \subseteq \mathrm{Ker}(\mathcal{K}_{\mathrm{Lie}(F)})$, cf. (iii). If $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect, then $\mathrm{Ker}(\mathrm{Lie}(F) \to \mathrm{Lie}(F^{\mathrm{ad}})) = 0$ and therefore $\mathrm{Lie}(F) = \mathrm{Lie}(F^{\mathrm{ad}})$. Thus to prove (a) we can assume that $F = F^{\mathrm{ad}}$ is simple; let $\mathcal{L}$ be the Lie type of F. If $\mathcal{L}$ is not of classical Lie type, then $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect if and only if either $p > 5$ or $p = 5$ and $\mathcal{L} \neq E_8$ (cf. [Hu2, TABLE, p. 49]). Thus to prove (a), we can assume that $\mathcal{L}$ is a classical Lie type. We fix a morphism $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(W(k))$.

Suppose that $\mathcal{L}$ is either A_n or C_n. By the standard trace form on $\mathrm{Lie}(F^{\mathrm{sc}})$ (resp. on $\mathrm{Lie}(F^{\mathrm{sc}}_{W(k)})$ or on $\mathrm{Lie}(F^{\mathrm{sc}}_{\mathbb{C}})$) we mean the trace form $\mathcal{T}$ (resp. $\mathcal{T}_{W(k)}$ or $\mathcal{T}_{\mathbb{C}}$) associated to the faithful representation of F^{sc} (resp. $F^{\mathrm{sc}}_{W(k)}$ or $F^{\mathrm{sc}}_{\mathbb{C}}$) of rank $n+1$ if $\mathcal{L} = A_n$ and of rank $2n$ if $\mathcal{L} = C_n$. We have $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}}_{\mathbb{C}})} = 2(n+1)\mathcal{T}_{\mathbb{C}}$, cf. [He, Ch. III, §8, (5) and (22)]. This identity implies that we also have $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}}_{W(k)})} = 2(n+1)\mathcal{T}_{W(k)}$ and $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}})} = 2(n+1)\mathcal{T}$. If p does not divide $2(n+1)$, then $\mathrm{Lie}(F^{\mathrm{sc}}) = \mathrm{Lie}(F)$ and it is well known that $\mathcal{T}$ is perfect; thus $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}})} = \mathcal{K}_{\mathrm{Lie}(F)} = 2(n+1)\mathcal{T}$ is perfect. Suppose that p divides $2(n+1)$. This implies that $\mathcal{K}_{\mathrm{Lie}(F^{\mathrm{sc}})}$ is the trivial bilinear form on $\mathrm{Lie}(F^{\mathrm{sc}})$. From this and (iv) we get that the restriction of $\mathcal{K}_{\mathrm{Lie}(F)}$ to $\mathrm{Im}(\mathrm{Lie}(F^{\mathrm{sc}}) \to \mathrm{Lie}(F))$ is trivial. As $\dim_k(\mathrm{Lie}(F)/\mathrm{Im}(\mathrm{Lie}(F^{\mathrm{sc}}) \to \mathrm{Lie}(F))) = 1$ and as $\dim_k(\mathrm{Lie}(F)) \geq 3$, we get that $\mathcal{K}_{\mathrm{Lie}(F)}$ is degenerate.

Suppose that $\mathcal{L} = B_n$ (resp. that $\mathcal{L} = D_n$ with $n \geq 4$). If $p > 2$ we have $\mathrm{Lie}(F^{\mathrm{sc}}) = \mathrm{Lie}(F)$. Moreover, using [He, Ch. III, §8, (11) and (15)], as in the previous paragraph we argue that $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect if p does not divide $2(2n-1)$ (resp. if p does not divide $2(n-1)$) and is degenerate if p divides $2n-1$ (resp. if p divides $2(n-1)$).

We are left to show that $\mathcal{K}_{\mathrm{Lie}(F)}$ is degenerate if $p = 2$ and $\mathcal{L} = B_n$. The group F is (isomorphic to) the SO-group of the quadratic form $x_0^2 + x_1x_{n+1} + \cdots + x_nx_{2n}$ on $W := k^{2n+1}$. Let $\{e_{i,j}|i,j \in \{0,1,\ldots,n\}$ be the standard k-basis for $\mathfrak{gl}_W$. The direct sum $\mathfrak{n}_n := \oplus_{i=1}^{2n} ke_{0,i}$ is a nilpotent ideal of $\mathrm{Lie}(F)$, cf. [Bo, Subsection 23.6]. Thus $\mathfrak{n}_n \subseteq \mathrm{Ker}(\mathcal{K}_{\mathrm{Lie}(F)})$, cf. [Bou1, Ch. I, §4, Prop. 6 (b)] applied to the adjoint representation of $\mathrm{Lie}(F)$. Therefore $\mathcal{K}_{\mathrm{Lie}(F)}$ is degenerate.

We conclude that $\mathcal{K}_{\mathrm{Lie}(F)}$ is perfect if and only if both conditions (i) and (ii) hold. Therefore (a) (and so also (b)) holds. $\square$

3.7. Remarks. **(a)** Let $p \in \mathbb{N}^*$ be a prime. Suppose that A is an algebraically closed field of characteristic p. Let G be an adjoint group over $\mathrm{Spec}(A)$ such that $\mathfrak{g} = \mathrm{Lie}(G)$,

cf. Theorem 3.5. We have $p \neq 2$, cf. Proposition 3.6. Let $G_{\mathbb{Z}}$ be the unique (up to isomorphism) split, adjoint group scheme over $\mathrm{Spec}(\mathbb{Z})$ such that G is the pull back of $G_{\mathbb{Z}}$ to $\mathrm{Spec}(A)$, cf. [DG, Vol. III, Exp. XXIII, Thm. 4.1, and Exp. XXV, Thm. 1.1]. We have $\mathfrak{g} = \mathrm{Lie}(G_{\mathbb{Z}}) \otimes_{\mathbb{Z}} A$ i.e., $\mathfrak{g}$ has a canonical model $\mathrm{Lie}(G_{\mathbb{Z}})$ over $\mathbb{Z}$. For $p > 7$, this result was obtained in [Cu, §5, Thm.]. For $p > 3$, this result was obtained by Seligman, Mills, Block, and Zassenhaus (see [BZ] and [Se, II. 10]). For $p = 3$, this result was obtained in [Br, Thm. 4.1]. It seems that the fact that $p \neq 2$ (i.e., that all Killing forms of finite dimensional Lie algebras over fields of characteristic 2, are degenerate) is new.

(b) Let $\mathrm{Sc\text{-}perf}_Y$ be the category defined in the same way as $\mathrm{Adj\text{-}perf}_Y$ but using simply connected semisimple group schemes over Y. Based on Theorem 3.5 and Lemma 3.6 (b), the natural functor $\mathrm{Sc\text{-}perf}_Y \to \mathrm{Lie\text{-}perf}_Y$ is also an equivalence of categories.

(c) Let $B \twoheadrightarrow A$ be an epimorphism of commutative $\mathbb{Z}$-algebras whose kernel J is a nilpotent ideal. Then $\mathfrak{g}$ has, up to isomorphisms, a unique lift to a Lie algebra over B which as a B-module is projective. One can prove this statement using cohomological methods as in the proof of Theorem 3.3. The statement also follows from Theorems 3.5 and the fact that $\mathrm{Aut}(\mathfrak{g})^0$ has, up to isomorphisms, a unique lift to an adjoint group scheme over $\mathrm{Spec}(B)$ (this can be easily checked at the level of torsors of adjoint group schemes; see [DG, Vol. III, Exp. XXIV, Cor. 1.17 and 1.18]).

3.8. Corollary. *The category $\mathrm{Lie\text{-}perf}_Y$ is non-empty if and only if Y is a non-empty $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2}])$-scheme.*

Proof: The if part is implied by the fact that an $\mathfrak{sl}_2$ Lie algebra over $\mathcal{O}_Y$ has perfect Killing form. The only if part follows from the relation $p \neq 2$ of the Remark 3.7 (a). $\square$

4. Proof of the Basic Theorem A

In this Section we prove the Basic Theorem A. See Subsections 4.1, 4.2, and 4.3 for the proofs of Theorems 1.1 (a), 1.1 (b), and 1.1 (c) (respectively). In Remarks 4.4 we point out that the hypotheses of Theorem 1.1 (b) and (c) are indeed needed in general. We will use the notations listed in Section 1.

4.1. Proof of 1.1 (a). Let $\mathrm{Aut}(\mathfrak{g})$ be the group scheme of automorphisms of $\mathfrak{g}$. Let $\mathrm{Aut}(\mathfrak{g})^0$ be the unique open subgroup scheme of $\mathrm{Aut}(\mathfrak{g})$ whose fibres are connected and which is an adjoint group scheme, cf. property 3.5 (i). The adjoint representation $\mathrm{Ad} : G_K \to GL_{\mathfrak{g}\otimes_A K}$ is a closed embedding (cf. Lemma 2.5.1) and it obviously factors through $\mathrm{Aut}(\mathfrak{g})_K$. As the fibres of G_K are connected, Ad factors through $\mathrm{Aut}(\mathfrak{g})^0_K$. Thus we have a natural homomorphism $G_K \to \mathrm{Aut}(\mathfrak{g})^0_K$ which is a closed embedding. As $\mathrm{ad}(\mathfrak{g}) = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0)$ (cf. property 3.5 (i)), we can identify $\mathrm{Lie}(G_K) = \mathfrak{g} \otimes_A K = \mathrm{ad}(\mathfrak{g}) \otimes_A K = \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g})^0_K)$. From this and the fact that G_K and $\mathrm{Aut}(\mathfrak{g})^0_K$ are smooth over $\mathrm{Spec}(K)$ and have connected fibres, we get that the homomorphism $G_K \to \mathrm{Aut}(\mathfrak{g})^0_K$ is also étale. The closed embedding, étale homomorphism $G_K \to \mathrm{Aut}(\mathfrak{g})^0_K$ is an isomorphism. Thus the adjoint group scheme $G := \mathrm{Aut}(\mathfrak{g})^0$ extends G_K and we have identities $\mathfrak{g} = \mathrm{ad}(\mathfrak{g}) = \mathrm{Lie}(G)$ that extend the identities $\mathfrak{g} \otimes_A K = \mathrm{ad}(\mathfrak{g}) \otimes_A K = \mathrm{Lie}(G_K)$. The uniqueness of G follows from either Theorem 3.5 or its construction. This ends the proof of Theorem 1.1 (b). $\square$

4.2. Proof of 1.1 (b). To prove Theorem 1.1 (b) and (c) we can assume Y is also integral. Let $K := K_Y$; it is a field. If F is a reductive group scheme over Y, then we have $\mathcal{D}(F) = \mathcal{D}(F_U)$ and thus the uniqueness parts of Theorem 1.1 (b) and (c) follow from Proposition 2.2 (b). We prove Theorem 1.1 (b). Due to the uniqueness part, to prove Theorem 1.1 (b) we can assume $Y = \mathrm{Spec}(A)$ is also local and strictly henselian. Let $\mathfrak{g} := \mathfrak{l}(Y)$ be the Lie algebra over A of global sections of $\mathfrak{l}$.

As U is connected, based on [DG, Vol. III, Exp. XXII, Prop. 2.8] we can speak about the split, adjoint group scheme S over Y of the same Lie type $\mathcal{L}$ as all geometric fibres of G_U. Let $\mathfrak{s} := \mathrm{Lie}(S)$. Let $\mathrm{Aut}(S)$ be the group scheme over Y of automorphisms of S. We have a short exact sequence $0 \to S \to \mathrm{Aut}(S) \to C \to 0$, where C is a finite, étale, constant group scheme over Y (cf. [DG, Vol. III, Exp. XXIV, Thm. 1.3]). Let $\gamma \in H^1(U, \mathrm{Aut}(S)_U)$ be the class that defines G_U. We recall that $GL_{\mathfrak{g}}$ and $GL_{\mathfrak{s}}$ are the reductive group schemes over Y of linear automorphisms of $\mathfrak{g}$ and $\mathfrak{s}$ (respectively). The adjoint representations define closed embedding homomorphisms $j_U : G_U \hookrightarrow (GL_{\mathfrak{g}})_U$ and $i : S \hookrightarrow GL_{\mathfrak{s}}$ and moreover i extends naturally to a closed embedding homomorphism $\mathrm{Aut}(S) \hookrightarrow GL_{\mathfrak{s}}$, cf. Lemma 2.5.1. Let $\delta \in H^1(U, (GL_{\mathfrak{s}})_U)$ be the image of γ via $i_U : S_U \hookrightarrow (GL_{\mathfrak{s}})_U$. We recall that the quotient sheaf for the faithfully flat topology of Y of the action of S on $GL_{\mathfrak{s}}$ via right translations, is representable by an Y-scheme $S\backslash GL_{\mathfrak{s}}$ that is affine and that makes $GL_{\mathfrak{s}}$ to be a right torsor of S over $S\backslash GL_{\mathfrak{s}}$ (cf. [CTS, Cor. 6.12]). Thus $S\backslash GL_{\mathfrak{s}}$ is a smooth, affine Y-scheme. The finite, étale, constant group scheme C acts naturally on $S\backslash GL_{\mathfrak{s}}$ and this action is free (cf. Lemma 2.5.1). From [DG, Vol I, Exp. V, Thm. 4.1] we get that the quotient Y-scheme $C\backslash(S\backslash GL_{\mathfrak{s}})$ is affine and that the quotient epimorphism $S\backslash GL_{\mathfrak{s}} \twoheadrightarrow C\backslash(S\backslash GL_{\mathfrak{s}})$ is an étale cover. Thus $C\backslash(S\backslash GL_{\mathfrak{s}})$ is a smooth, affine scheme over Y that represents the quotient sheaf for the faithfully flat topology of Y of the action of $\mathrm{Aut}(S)$ on $GL_{\mathfrak{s}}$ via right translations. From constructions we get that $GL_{\mathfrak{s}}$ is a right torsor of $\mathrm{Aut}(S)$ over $\mathrm{Aut}(S)\backslash GL_{\mathfrak{s}} := C\backslash(S\backslash GL_{\mathfrak{s}})$.

The twist of the class i_U via γ is j_U. As the A-modules $\mathfrak{g}$ and $\mathfrak{s}$ are isomorphic (being free of equal ranks), the class δ is trivial. Thus γ is the coboundary of a class in $H^0(U, (\mathrm{Aut}(S)\backslash GL_{\mathfrak{s}})_U)$. But $H^0(U, (\mathrm{Aut}(S)\backslash GL_{\mathfrak{s}})_U) = H^0(Y, \mathrm{Aut}(S)\backslash GL_{\mathfrak{s}})$ (cf. Proposition 2.2 (b)) and thus γ is the restriction of a class in $H^1(Y, \mathrm{Aut}(S))$. As Y is strictly henselian, each class in $H^1(Y, \mathrm{Aut}(S))$ is trivial. Thus γ is the trivial class. Therefore $G_U \xrightarrow{\sim} S_U$ i.e., G_U extends to an adjoint group scheme G over Y isomorphic to S. This ends the proof of Theorem 1.1 (b). $\square$

4.3. Proof of 1.1 (c). Let K_1 be the smallest Galois extension of K such that $E^{\mathrm{ab}}_{K_1}$ is a split torus. Let Y_1 be the normalization of Y in K_1. Let $U_1 := Y_1 \times_Y U$. From Proposition 2.3 we get that the maximal open subscheme of U_1 that is étale over U, surjects onto U. Therefore, as K_1 is a Galois extension of K, the morphism $U_1 \to U$ is a Galois cover. As the pair $(Y, Y \setminus U)$ is quasi-pure, the morphism $Y_1 \to Y$ is a Galois cover. We identify $\mathrm{Aut}_Y(Y_1) = \mathrm{Gal}(K_1/K)$. As $E^{\mathrm{ab}}_{K_1}$ is a split torus, there exists $s \in \mathbb{N}$ such that we have an isomorphism $\mathbb{G}^s_{mK_1} \xrightarrow{\sim} E^{\mathrm{ab}}_{K_1}$. It extends to an isomorphism $\mathbb{G}^s_{mU_1} \xrightarrow{\sim} E^{\mathrm{ab}}_{U_1}$. [Argument: to check this we can work locally in the étale topology of U_1 and thus we can assume $E^{\mathrm{ab}}_{U_1}$ is split (cf. Proposition 2.3); but this case is trivial.] Thus $E^{\mathrm{ab}}_{U_1}$ extends to a split torus $E^{\mathrm{ab}}_{Y_1}$ over Y_1 of rank s. We identity naturally E^{ab}_U with a closed subgroup scheme of $\mathrm{Res}_{U_1/U} E^{\mathrm{ab}}_{U_1}$, cf. [BLR, Ch. 7, 7.6, bottom of p. 197]. Let E^{ab} be the Zariski

closure of E^{ab}_U in $\mathrm{Res}_{Y_1/Y}E^{\mathrm{ab}}_{Y_1}$. We check that E^{ab} is a subtorus of $\mathrm{Res}_{Y_1/Y}E^{\mathrm{ab}}_{Y_1}$. It suffices to check that $E^{\mathrm{ab}}_{Y_1}$ is a subtorus of $(\mathrm{Res}_{Y_1/Y}E^{\mathrm{ab}}_{Y_1})_{Y_1} = \times_{a\in\mathrm{Aut}_Y(Y_1)}E^{\mathrm{ab}}_{Y_1}\times_{Y_1\ a}Y_1$. But the embedding $E^{\mathrm{ab}}_{Y_1} \to \times_{a\in\mathrm{Aut}_Y(Y_1)}E^{\mathrm{ab}}_{Y_1}\times_{Y_1\ a}Y_1$ is isomorphic to the natural embedding $\mathbb{G}^s_{mY_1} \hookrightarrow \times_{a\in\mathrm{Aut}_Y(Y_1)}\mathbb{G}^s_{mY_1}\times_{Y_1\ a}Y_1$ (i.e., with the diagonal embedding $\mathbb{G}^s_{mY_1} \to \mathbb{G}^{s[K_1:K]}_{mY_1}$). Thus $E^{\mathrm{ab}}_{Y_1}$ is a subtorus of $(\mathrm{Res}_{Y_1/Y}E^{\mathrm{ab}}_{Y_1})_{Y_1}$. Thus E^{ab} is a torus over Y that extends E^{ab}_U.

Let E^{ad} be the adjoint group scheme over Y that extends E^{ad}_U, cf. Theorem 1.1 (b). Let $F \to E^{\mathrm{ad}}\times_Y E^{\mathrm{ab}}$ be the central isogeny over Y that extends the central isogeny $E_K \to E^{\mathrm{ad}}_K \times_{\mathrm{Spec}(K)} E^{\mathrm{ab}}_K$, cf. Lemma 2.3.1 (a). Both F_U and E_U are the normalization of $E^{\mathrm{ab}}_U \times_U E^{\mathrm{ad}}_U$ in E_K, cf. Lemma 2.3.1 (b). Thus $E_U = F_U$ extends uniquely to a reductive group scheme $E := F$ over Y (cf. the first paragraph of Subsection 4.2 for the uniqueness part). This ends the proof of Theorem 1.1 (c) and thus also of the Basic Theorem A. $\square$

4.4. Remarks. (a) Let $Y_1 \to Y$ be a finite, non-étale morphism between normal, noetherian, integral $\mathrm{Spec}(\mathbb{Z}_{(2)})$-schemes such that there exists an open subscheme U of Y with the properties that: (i) $Y\setminus U$ has codimension in Y at least 2, and (ii) $Y_1\times_Y U \to U$ is a Galois cover of degree 2. Let E_U be the rank 1 non-split torus over U that splits over $Y_1 \times_Y U$. Then E_U does not extend to a smooth, affine group scheme over Y. If moreover $Y = \mathrm{Spec}(A)$ is an affine $\mathrm{Spec}(\mathbb{F}_2)$-scheme, then we have $Lie(E_U)(U) = A$ and therefore $Lie(E_U)$ extends to a Lie algebra over $\mathcal{O}_Y$ which as an $\mathcal{O}_Y$-module is free. Thus the quasi-pure part of the hypotheses of Theorem 1.1 (c) is needed in general.

(b) Suppose that $Y = \mathrm{Spec}(A)$ is local, strictly henselian, regular, and of dimension $d\geq 3$. Let $K := K_Y$. Let $m \in \mathbb{N}^*$ be such that there exists an A-submodule M of K^m that contains A^m, that is of finite type, that is not free, and that satisfies the identity $M = \cap_{V\in\mathcal{D}(Y)} M\otimes_A V$ (inside $M\otimes_A K$). A typical example (communicated to us by Serre): $m = d-1$ and $M \xrightarrow{\sim} \mathrm{Coker}(f)$, where the A-linear map $f : A \to A^d$ takes 1 into a d-tuple $(x_1,\ldots,x_d)$ of regular parameters of A.

Let $\mathcal{F}$ be the coherent $\mathcal{O}_Y$-module defined by M. Let U be an open subscheme of Y such that $Y\setminus U$ has codimension in Y at least 2 and the restriction $\mathcal{F}_U$ of $\mathcal{F}$ to U is a locally free $\mathcal{O}_U$-module. Let E_U be the reductive group scheme over U of linear automorphisms of $\mathcal{F}_U$. We sketch the reason why the assumption that E_U extends to a reductive group scheme E over Y leads to a contradiction. The group scheme E is isomorphic to GL_{mA} (as A is strictly henselian) and therefore there exists a free A-submodule N of K^m of rank m such that we can identify $E = GL_N$. As A is a unique factorization domain (being local and regular), it is easy to see that there exists an element $f\in K$ such that the identity $M\otimes_A V = fN\otimes_A V$ holds for each $V\in\mathcal{D}(Y)$. This implies that $M = fN$. Thus M is a free A-module. Contradiction.

As E_U does not extend to a reductive group scheme over Y and as the pair $(Y, Y\setminus U)$ is quasi-pure, from Subsection 4.3 we get that E^{ad}_U also does not extend to an adjoint group scheme over Y. Therefore the Lie part of the hypotheses of Theorem 1.1 (b) is needed in general.

5. Extending homomorphisms via Zariski closures

In this Section we prove three results on extending homomorphisms of reductive group schemes via taking (normalizations of) Zariski closures. The first one complements

Theorem 1.1 (c) and Proposition 2.5.2 (see Proposition 5.1) and the other two refine parts of [Va1] (see Subsections 5.2 to 5.5).

5.1. Proposition. *Let $Y = Spec(R)$ be a local, regular scheme of dimension 2. Let y be the closed point of Y. Let k be the residue field of y. Let $U := Y \setminus \{y\}$. Let $K := K_Y$. Let $m \in \mathbb{N}^*$. Let E_K be a closed subgroup scheme of GL_{mK}. Let E_U be the normalization of the Zariski closure of E_K in GL_{mU}. Let $\rho_U : E_U \to GL_{mU}$ be the resulting homomorphism. Suppose that E_U is a reductive group scheme over U. We have:*

(a) *The homomorphism ρ_U extends uniquely to a finite homomorphism $\rho : E \to GL_{mY}$ between reductive group schemes over Y.*

(b) *If $char(k) = 2$, we assume that E_K has no normal subgroup that is adjoint of isotypic B_n Dynkin type for some $n \in \mathbb{N}^*$. Then ρ is a closed embedding.*

Proof: As the pair $(Y, Y \setminus U)$ is quasi-pure (see Section 1) and as the Lie algebra Lie(E_U) over $\mathcal{O}_U$ extends to a Lie algebra over $\mathcal{O}_Y$ that is locally a free $\mathcal{O}_Y$-module (cf. Proposition 2.2 (a)), the hypotheses of Theorem 1.1 (c) hold and thus from Theorem 1.1 (c) we get that there exists a unique reductive group scheme E over Y that extends E_U. We write $E = \text{Spec}(R_E)$ and $GL_{mY} = \text{Spec}(R_{GL_{mY}})$. As $\mathcal{D}(E) = \mathcal{D}(E_U)$ and as $\mathcal{D}(GL_{mY}) = \mathcal{D}(GL_{mU})$, from Proposition 2.2 (a) we get that R_E and $R_{GL_{mY}}$ are the R-algebras of global functions of E_U and GL_{mU} (respectively). Let $R_{GL_{mY}} \to R_E$ be the R-homomorphism defined by ρ_U and let $\rho : E \to GL_{\mathcal{F}}$ be the morphism of Y-schemes it defines. The morphism ρ is a homomorphism as it is so generically. To check that ρ is finite, we can assume that R is complete. Thus R_E and $R_{GL_{mY}}$ are excellent rings, cf. [Ma, §34]. Therefore the normalization $E' = \text{Spec}(R_{E'})$ of the Zariski closure of E_K in GL_{mY} is a finite, normal GL_{mY}-scheme.

The identity components of the reduced geometric fibres of ρ are trivial groups, cf. Proposition 2.5.2 (a) or (b). Thus ρ is a quasi-finite morphism. From Zariski Main Theorem (see [Gr1, Thm. (8.12.6)]) we get that E is an open subscheme of E'. But from Proposition 2.5.2 (b) we get that the morphism $E \to E'$ satisfies the valuative criterion of properness with respect to discrete valuation rings whose fields of fractions are K. As each local ring of E' is dominated by a discrete valuation ring of K, we get that the morphism $E \to E'$ is surjective. Thus the open, surjective morphism $E \to E'$ is an isomorphism. Thus ρ is finite i.e., (a) holds.

We prove (b). The pull back of ρ via a dominant morphism $\text{Spec}(V) \to Y$, with V a discrete valuation ring, is a closed embedding (cf. Proposition 2.5.2 (c)). This implies that the fibres of ρ are closed embeddings. Thus q is a closed embedding, cf. Theorem 2.5. $\square$

We have the following refinement of [Va1, Lemma 3.1.6].

5.2. Proposition. *Let $Y = Spec(A)$ be a reduced, affine scheme. Let K be a localization of A. Let m, $s \in \mathbb{N}^*$. For $j \in \{1, \ldots, s\}$ let G_{jK} be a reductive, closed subgroup scheme of GL_{mK}. We assume that the group subschemes G_{jK}'s commute among themselves and that one of the following two things holds:*

(i) *either the direct sum $\oplus_{j=1}^{s} Lie(G_{jK})$ is a Lie subalgebra of $Lie(GL_{mK})$, or*

(ii) *$s = 2$, G_{1K} is a torus, and G_{2K} is a semisimple group scheme.*

Then the closed subgroup scheme G_{0K} of GL_{mK} generated by G_{jK}'s exists and is reductive. Moreover, we have:

(a) *If (i) holds, then $Lie(G_{0K}) = \oplus_{j=1}^{s} Lie(G_{jK})$.*

(b) *We assume that for each $j \in \{1, \ldots, s\}$ the Zariski closure G_j of G_{jK} in GL_{mA} is a reductive group scheme over Y. Then the Zariski closure G_0 of G_{0K} in GL_{mA} is a reductive, closed subgroup scheme of GL_{mA}.*

Proof: Let Λ be the category whose objects $Ob(\Lambda)$ are finite subsets of K and whose morphisms are the inclusions of subsets. For $\alpha \in Ob(\Lambda)$, let K_α be the $\mathbb{Z}$-subalgebra of K generated by α and let $A_\alpha = A \cap K_\alpha$. We have $K = \mathrm{ind.lim.}_{\alpha \in Ob(\Lambda)} K_\alpha$ and $A = \mathrm{ind.lim.}_{\alpha \in Ob(\Lambda)} K_\alpha$. The reductive group schemes G_{jK} are of finite presentation. Based on this and [Gr1, Thms. (8.8.2) and (8.10.5)], one gets that there there exists $\beta \in Ob(\Lambda)$ such that each G_{jK} is the pull back of a closed subgroup scheme of GL_{mK_β}. For $\alpha \supseteq \beta$, the set $C(\alpha)$ of points of $\mathrm{Spec}(K_\alpha)$ with the property that the fibres over them of all morphisms $G_{jK_\alpha} \to \mathrm{Spec}(K_\alpha)$ are (geometrically) connected, is a constructible set (cf. [Gr1, Thm. (9.7.7)]). We have $\mathrm{proj.lim.}_{\alpha \in Ob(\Lambda)} C(\alpha) = \mathrm{Spec}(K)$. From this and [Gr1, Thm. (8.5.2)], we get that there exists $\beta_1 \in Ob(\Lambda)$ such that $\beta_1 \supseteq \beta$ and $C(\beta_1) = \mathrm{Spec}(K_{\beta_1})$. Thus by replacing β with β_1, we can assume that the fibres of all morphisms $G_{jK_\beta} \to \mathrm{Spec}(K_\beta)$ are connected. A similar argument shows that, by enlarging β, we can assume that all morphisms $G_{jK_\beta} \to \mathrm{Spec}(K_\beta)$ are smooth and their fibres are reductive groups (the role of [Gr1, Thm. (9.7.7)] being replaced by [Gr1, Prop. (9.9.5)] applied to the $\mathcal{O}_{G_{jK_\alpha}}$-module $Lie(G_{jK_\alpha})$ and respectively by [DG, Vol. III, Exp. XIX, Cor. 2.6]). Thus each G_{jK_β} is a reductive closed subgroup scheme of GL_{mK_β}. The smooth group schemes G_{jK_β}'s commute among themselves as this is so after pull back through the dominant morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(K_\beta)$. By enlarging β, we can also assume that either (i) or (ii) holds for them and that K_β is a localization of A_β. By replacing A with the local ring of $\mathrm{Spec}(A_\beta)$ dominated by A, we can assume that A is a localization of a reduced, finitely generated $\mathbb{Z}$-algebra.

Using induction on $s \in \mathbb{N}^*$, it suffices to prove the Proposition for $s = 2$. Moreover, we can assume that $K = K_Y$. For the sake of flexibility, in what follows we will only assume that A is a reduced, noetherian $\mathbb{Z}$-algebra; thus K is a finite product of fields. As all the statements of the Proposition are local for the étale topology of Y, it suffices to prove the Proposition under the extra assumption that G_1 and G_2 are split (cf. Proposition 2.3). Let $C_K := G_{1K} \cap G_{2K}$. It is a closed subgroup scheme of G_{jK} that commutes with G_{jK}, $j \in \{1, 2\}$. The Lie algebra $\mathrm{Lie}(C_K)$ is included in $\mathrm{Lie}(G_{1K}) \cap \mathrm{Lie}(G_{2K})$ and therefore it is trivial if (i) holds. Thus if (i) holds, then C_K is a finite, étale, closed subgroup scheme of $Z(G_{jK})$. If (ii) holds, then C_K is a closed subgroup scheme of $Z(G_{2K})$ and thus (as K is a finite product of fields) it is a finite group scheme of multiplicative type.

Let C be the Zariski closure of C_K in GL_{mA}. It is a closed subscheme of any maximal torus T_j of G_j. Let $T_1 \times_Y T_2 \to GL_{mA}$ be the product homomorphism. Its kernel $\mathfrak{K}$ is a group scheme over Y of multiplicative type (cf. Lemma 2.3.2 (a)) isomorphic to $T_1 \cap T_2$. But $\mathfrak{K}_K \xrightarrow{\sim} C_K$ is a finite group scheme over $\mathrm{Spec}(K)$ and therefore $\mathfrak{K}$ is a finite, flat group scheme over Y of multiplicative type (cf. Lemma 2.3.2 (b)). Thus $T_1 \cap T_2$ is a finite, flat group scheme over Y and therefore it is equal to C. We embed C in $G_1 \times_Y G_2$ via the

natural embedding $C \hookrightarrow G_1$ and via the composite of the inverse isomorphism $C \xrightarrow{\sim} C$ with the natural embedding $C \hookrightarrow G_2$. Let $G_{1,2} := (G_1 \times_Y G_2)/C$; it is a reductive group scheme over Y. We have a natural product homomorphism $q : G_{1,2} \to GL_{mA}$ whose pull back to $\mathrm{Spec}(K)$ can be identified with the closed embedding homomorphism $G_{0K} \hookrightarrow GL_{mK}$. Therefore G_{0K} is a reductive group scheme over $\mathrm{Spec}(K)$. Moreover, if (i) holds, then as C_K is étale we have natural identities $\mathrm{Lie}(G_{1K}) \oplus \mathrm{Lie}(G_{2K}) = \mathrm{Lie}(G_{1,2K}) = \mathrm{Lie}(G_{0K})$. Thus (a) holds. If q is a closed embedding, then q induces an isomorphism $G_{1,2} \xrightarrow{\sim} G_0$ and therefore G_0 is a reductive, closed subgroup scheme of GL_{mA}. Thus to end the proof of (b), we only have to show that q is a closed embedding.

To check that q is a closed embedding, it suffices to check that the fibres of q are closed embeddings (cf. Theorem 2.5). For this we can assume A is a complete discrete valuation ring which has an algebraically closed residue field k; this implies that G_0 is a flat, closed subgroup scheme of GL_{mA}. Let $\mathfrak{n} := \mathrm{Lie}(\mathrm{Ker}(q_k))$. From Proposition 2.5.2 (a) and Lemma 2.4 we get that: either (iii) $\mathfrak{n} = 0$ or (iv) $\mathrm{char}(k) = 2$ and there exists a normal subgroup F_k of $G_{1,2k}$ which is isomorphic to SO_{2n+1k} for some $n \in \mathbb{N}^*$ and for which we have $\mathrm{Lie}(F_k) \cap \mathfrak{n} \neq 0$. We show that the assumption that (iv) holds leads to a contradiction. Let F be a normal, closed subgroup scheme of $G_{1,2}$ that lifts F_k and that is isomorphic to SO_{2n+1A} (cf. last paragraph of the proof of Proposition 2.5.2 (c)). Let $j_0 \in \{1, 2\}$ be such that $F \lhd G_{j_0} \lhd G_{1,2}$ (if (ii) holds, then $j_0 = 2$). As G_{j_0} is a closed subgroup scheme of GL_{mA}, we have $\mathrm{Lie}(G_{j_0k}) \cap \mathfrak{n} = 0$ and therefore also $\mathrm{Lie}(F_k) \cap \mathfrak{n} = 0$. Contradiction. Thus (iv) does not hold and therefore (iii) holds. Thus $\mathrm{Ker}(q_k)$ is a finite, étale, normal subgroup of $G_{1,2k}$. Thus $\mathrm{Ker}(q_k)$ is a subgroup of $Z(G_{1,2k})$ and therefore also of any maximal torus of $G_{1,2k}$. From this and Proposition 2.5.2 (a) we get that $\mathrm{Ker}(q_k)$ is trivial. Therefore q_k is a closed embedding. Thus q is a closed embedding. $\square$

In the last part of Section 5 we present significant refinements and simplifications to the fundamental results [Va1, Prop. 4.3 10 and Rm. 4.3.10.1 1)].

5.3. Basic Theorem B. *Let $Y = Spec(A)$ be a local, integral scheme. Let $K := K_Y$ and let k be the residue field of the closed point y of Y. Let M be a free A-module of finite rank. Let G_K be a reductive, closed subgroup scheme of $GL_{M\otimes_A K}$ and let $\tilde{T}_K$ be a maximal torus of G_K. Let $\mathfrak{h} := Lie(G_K^{\mathrm{der}}) \cap \mathfrak{gl}_M$, the intersection being taken inside $\mathfrak{gl}_M \otimes_A K$. Let $Z^0(G)$, G^{der}, and G be the Zariski closures of $Z^0(G_K)$, G_K^{der}, and G_K (respectively) in GL_M. We assume that the following five conditions hold:*

(i) *$Z^0(G)$ is a closed subgroup scheme of GL_M that is a torus;*

(ii) *the Lie algebra $\mathfrak{h}$ is a direct summand of $\mathfrak{gl}_M$ and there exists a semisimple group scheme H over Y which extends G_K^{der} and for which we have an identity $Lie(H) = \mathfrak{h}$ that extends the identities $Lie(H_K) = Lie(G_K^{\mathrm{der}}) = \mathfrak{h} \otimes_A K$;*

(iii) *if Y is not normal and if $char(k)$ is a prime $p \in \mathbb{N}^*$, then for each one dimensional $\bar{K}$-vector subspace $\mathfrak{n}$ of $Lie(G_{\bar{K}}^{\mathrm{der}}) = \mathfrak{h} \otimes_A \bar{K}$ on which $\tilde{T}_{\bar{K}}$ acts via a non-trivial character of $\tilde{T}_{\bar{K}}$, the order of nilpotency of every endomorphism $u \in \mathfrak{n} \subseteq End_{\bar{K}}(M \otimes_A \bar{K})$ is at most p and moreover the exponential map $Exp_{\mathfrak{n}} : \mathfrak{n} \to GL_{M\otimes_A \bar{K}}(\bar{K})$ that takes $u \in \mathfrak{n}$ into $Exp_{\mathfrak{n}}(u) := \sum_{l=0}^{p-1} \frac{u^l}{l!}$, factors through the group of $\bar{K}$-valued points of a $\mathbb{G}_{a\bar{K}}$-subgroup of $G_{\bar{K}}^{\mathrm{der}} = H_{\bar{K}}$ normalized by $\tilde{T}_{\bar{K}}$;*

(iv) *there exists a field extension $\tilde{K}$ of K such that $\tilde{T}_{\tilde{K}}$ is split and we have a finite family $(\tilde{T}_{i\tilde{K}})_{i\in I}$ of rank 1 subtori of $\tilde{T}_{\tilde{K}}$ which are equipped with isomorphisms $i_{\tilde{T}_{i\tilde{K}}}: \tilde{T}_{i\tilde{K}} \xrightarrow{\sim} \mathbb{G}_{m\tilde{K}}$ and for which the following four things hold:*

(iv.a) *the torus $\tilde{T}_{\tilde{K}}$ is generated by $Z^0(G_K)_{\tilde{K}}$ and by the $\tilde{T}_{i\tilde{K}}$'s ($i \in I$);*

(iv.b) *for each $i \in I$ we have a direct sum decomposition $M \otimes_A \tilde{K} = \oplus_{\gamma_i \in \mathbb{S}_i} M_{i\gamma_i\tilde{K}}$ in $\tilde{K}$-vector spaces such that $\tilde{T}_{i\tilde{K}}$ acts on $M_{i\gamma_i\tilde{K}}$ via the diagonal character γ_i of $\tilde{T}_{i\tilde{K}}$ with respect to $i_{\tilde{T}_{i\tilde{K}}}$ (thus $\mathbb{S}_i$ is a finite subset of the group of diagonal characters of $\tilde{T}_{i\tilde{K}}$ with respect to $i_{\tilde{T}_{i\tilde{K}}}$, viewed additively and identified with $\mathbb{Z}$);*

(iv.c) *if char(k) is a prime $p \in \mathbb{N}^*$, then for each $i \in I$ either:*

– the set $\mathbb{S}_i$ is of p-type 1 (in the sense of Definitions 2.6) or
– the set $\mathbb{S}_i$ is of p-type 2 (resp. p-type 3), and $\tilde{T}_{i\tilde{K}}$ is a torus of a semisimple subgroup $\tilde{S}_{i\tilde{K}}$ of $G^{\mathrm{der}}_{\tilde{K}}$ which is isomorphic to $SL_{2\tilde{K}}$ (resp. $PGL_{2\tilde{K}}$) and which is normalized by $\tilde{T}_{\tilde{K}}$;

(iv.d) *there exists an étale cover of $\tilde{T}_{i\tilde{K}}$ that has a degree $a_i \in \mathbb{N}^*$ which is not divisible by char(k) and that is naturally a torus of $Z^0(G_K)_{\tilde{K}} \times_{\mathrm{Spec}(\tilde{K})} H_{\tilde{K}}$;*

(v) *if Y is not normal, then $Z^0(G) \times_Y H$ splits over an integral, étale cover of Y.*

Then G (resp. G^{der}) is a reductive (resp. semisimple), closed subgroup scheme of GL_M.

Proof: We consider the following statement $\mathfrak{S}$: G^{der} is a semisimple, closed subgroup scheme of GL_M. The closed subgroup schemes $G_{1K} := Z^0(G_K)$ and $G_{2K} := G^{\mathrm{der}}_K$ of $GL_{M\otimes_A K}$ commute and condition 5.2 (ii) holds. Thus (cf. Proposition 5.2 and (i)), to prove the Theorem it suffices to show that the statement $\mathfrak{S}$ is true. The statement $\mathfrak{S}$ is local for the étale topology of Y. We recall that a connected, étale scheme over a normal scheme, is a normal, integral scheme. If Y is (resp. is not) normal, then from Proposition 2.3 and this recalled fact (resp. from (v)), we get that we can assume that H has a split, maximal torus T and that $Z^0(G)$ is a split torus. As G_{1K} and G_{2K} are of finite presentation over Spec(K) and as $\mathfrak{h}$ is a direct summand of $\mathfrak{gl}_M$, as in the first paragraph of the proof of Proposition 5.2 we argue that to prove the Theorem we can also assume A is a local ring of a finitely generated, integral $\mathbb{Z}$-algebra. Thus A is an excellent, local, integral ring (cf. [Ma, §34]); we emphasize that below we will use only these three properties of A.

Let $U := Y \setminus \{y\}$. As the statement $\mathfrak{S}$ is local for the Zariski topology of Y, by localizing we can also assume that G^{der}_U is a semisimple, closed subgroup scheme of $(GL_M)_U$. Let $G' \to Z^0(G) \times_Y H^{\mathrm{ad}}$ be the central isogeny that extends the central isogeny $G_K \to Z^0(G_K) \times_{\mathrm{Spec}(K)} H^{\mathrm{ad}}_K$, cf. Lemma 2.3.1 (a). We can identify G'^{der} with H.

The two split, maximal tori $\tilde{T}_{\tilde{K}}$ and $\mathrm{Im}(Z^0(G_{\tilde{K}}) \times_{\mathrm{Spec}(\tilde{K})} T_{\tilde{K}} \to G_{\tilde{K}})$ of $G_{\tilde{K}}$ are $G^{\mathrm{ad}}_{\tilde{K}}(\tilde{K})$-conjugate (cf. [DG, Vol. III, Exp. XXIV, Lemma 1.5]) and thus, up to a replacement of $\tilde{K}$ by a finite field extension of it, are also $G^{\mathrm{der}}_{\tilde{K}}(\tilde{K})$-conjugate. Thus up to such a replacement of $\tilde{K}$ and up to $G^{\mathrm{der}}_{\tilde{K}}(\tilde{K})$-conjugation, we can assume that the family of tori $(\tilde{T}_{i\tilde{K}})_{i\in I}$ is the pull back of a family $(\tilde{T}_{iK})_{i\in I}$ of subtori of $\tilde{T}_K$ and that $\tilde{T}_K$ is generated by T_K and $Z^0(G_K)$. Thus for each $i \in I$ there exists a $\mathbb{G}_{mK}$ subgroup T_{iK} of $Z^0(G_K) \times_{\mathrm{Spec}(K)} T_K$ which (cf. (iv.d)) is an étale cover of $\tilde{T}_{iK}$ of some degree $a_i \in \mathbb{N}^*$ that

is not divisible by $\mathrm{char}(k)$. The Zariski closure T_i of T_{iK} in the split torus $Z^0(G) \times_Y T$ is a split torus over Y of rank 1. Let $\tilde{T}_i$ be the image of T_i in G' via the central isogeny $Z^0(G) \times_Y H \to G'$; it is a split torus over Y that has rank 1 and that extends $\tilde{T}_{iK}$. As $\tilde{K}$ is a field, the isomorphism $i_{\tilde{T}_{i\tilde{K}}} : \tilde{T}_{i\tilde{K}} \tilde{\to} \mathbb{G}_{m\tilde{K}}$ is the pull back of an isomorphism $i_{\tilde{T}_i} : \tilde{T}_i \tilde{\to} \mathbb{G}_{mY}$. We identify $\mathcal{S}_i$ with a set of diagonal characters of $\tilde{T}_{iK}$ with respect to $i_{\tilde{T}_i} \times_Y \mathrm{Spec}(K)$. The direct sum decomposition $M \otimes_A \tilde{K} = \oplus_{\gamma_i \in \mathcal{S}_i} M_{i\gamma_i\tilde{K}}$ of (iv.b) is the tensorization with $\tilde{K}$ of a direct sum decomposition $M \otimes_A K = \oplus_{\gamma_i \in \mathcal{S}_i} M_{i\gamma_i K}$.

If we are in the case of (iv.c) when $\tilde{T}_{i\tilde{K}}$ is a torus of $\tilde{S}_{i\tilde{K}}$, then $T_i = \tilde{T}_i$ is a torus of a semisimple, closed subgroup scheme S_i of H which is normalized by T, whose adjoint is isomorphic to PGL_{2Y}, and whose pull back to $\mathrm{Spec}(\tilde{K})$ is $\tilde{S}_{i\tilde{K}}$. [Argument: If Φ is the root system of $\mathfrak{h} = \mathrm{Lie}(H)$ with respect to the split, maximal torus T of H, then to $\tilde{S}_{i\tilde{K}}$ corresponds naturally a closed subset $\tilde{\Phi}_i$ of Φ of the form $\{\tilde{\alpha}_i, -\tilde{\alpha}_i\}$ with $\tilde{\alpha}_i \in \Phi$; thus the existence of S_i is implied by [DG, Vol. III, Exp. XXII, Thm. 5.4.7 and Prop. 5.10.1].]

Let $s_i \in \mathrm{Lie}(T_i)$ be the standard generator. The element $\frac{1}{a_i} s_i \in \mathrm{Lie}(\tilde{T}_i)$ is the standard generator of $\mathrm{Lie}(\tilde{T}_i)$ and it is a semisimple element of $\mathrm{Lie}(Z^0(G)) \oplus \mathrm{Lie}(H) = \mathrm{Lie}(Z^0(G)) \oplus \mathfrak{h}$ and thus also of $\mathfrak{gl}_M$. Thus M is a left $\mathrm{Lie}(\tilde{T}_i)$-module. We check that the closed embedding homomorphism $\tilde{T}_{iK} \hookrightarrow GL_{M \otimes_A K}$ extends to a closed embedding homomorphism $\tilde{T}_i \to GL_M$. The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 1 follows from Proposition 2.6.2 applied with $(T, i_T) = (\tilde{T}_i, i_{\tilde{T}_i})$, cf. (iv.b) and (iv.c). The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 2 follows from Proposition 2.6.2 applied with $(G, T, i_T) = (S_i, T_i = \tilde{T}_i, i_{\tilde{T}_i})$, cf. (iv.b) and (iv.c). The case when $\mathrm{char}(k) = p$ and $\mathcal{S}_i$ is of p-type 3 follows from Proposition 2.6.2 applied with (G, T, i_T) as $(S_i', T_i' = \tilde{T}_i', i_{\tilde{T}_i'})$, where S_i' is the simply connected semisimple group scheme cover of S_i, where $\tilde{T}_i'$ is the inverse image of $\tilde{T}_i$ to S_i' and where $i_{\tilde{T}_i'} : \tilde{T}_i' \tilde{\to} \mathbb{G}_{mY}$ is the natural isomorphism that is compatible with $i_{\tilde{T}_i}$ (cf. (iv.b) and (iv.c)); we add that the degree of the isogeny $\tilde{T}_i' \to \tilde{T}_i$ is 2 and thus the set of characters of the action of $\tilde{T}_i'$ on $M \otimes_A K$ is naturally identified with $\frac{1}{2}\mathcal{S}_i$ and therefore (as $\mathcal{S}_i$ it is of p-type 3) it is of p-type 2. The case when $\mathrm{char}(k) = 0$ is well known. [Argument: we have a direct sum decomposition $M = \oplus_{\gamma_i \in \mathcal{S}_i} M_{i\gamma_i}$ (automatically in free A-modules), where $M_{i\gamma_i} := M \cap M_{i\gamma_i K}$; thus M is naturally a left $\tilde{T}_i$-module and from Lemma 2.3.2 (b) and (c) we get that the homomorphism $\tilde{T}_i \to GL_M$ is a closed embedding].

Let $h : Z^0(G) \times_Y \times_{i \in I} \tilde{T}_i \to GL_M$ be the natural product homomorphism. The kernel $\mathrm{Ker}(h)$ is a group scheme of multiplicative type, cf. Lemma 2.3.2 (a). Let

$$\tilde{T} := (Z^0(G) \times_Y \times_{i \in I} \tilde{T}_i)/\mathrm{Ker}(h);$$

it is a split torus over Y that lifts $\tilde{T}_K$, that is a torus of GL_M (cf. Lemma 2.3.2 (c)), and that has T as a subtorus. Thus we can identify naturally T with a split, maximal torus of H as well as with a torus of GL_M (or of $\tilde{T}$) contained in G^{der}.

The role of T is that of an arbitrary split, maximal torus of H. Thus for the rest of the proof that the statement $\mathfrak{S}$ holds, we will only use the following property:

(vi) *each split, maximal torus of H is naturally a torus of GL_M contained in G^{der}, the Lie algebra Lie(H) = $\mathfrak{h}$ is a direct summand of $\mathfrak{gl}_M$, G_U^{der} is a semisimple, closed subgroup scheme of $(GL_M)_U$, and the property (iii) holds.*

5.3.1. Notations. Let $\mathrm{Aut}(\mathfrak{h})$ be the group scheme of Lie automorphisms of $\mathfrak{h}$. The adjoint representation defines a closed embedding homomorphism $H^{\mathrm{ad}} \to \mathrm{Aut}(\mathfrak{h})$, cf. Lemma 2.5.1. As $\mathfrak{h}$ is a direct summand of the left GL_M-module $\mathfrak{gl}_M$, we can speak about the normalizer N of $\mathfrak{h}$ in GL_M. Obviously G^{der} is a closed subscheme of N and thus we have a natural morphism $G^{\mathrm{der}} \to \mathrm{Aut}(\mathfrak{h})$ that factors through the closed subscheme H^{ad} of $\mathrm{Aut}(\mathfrak{h})$ (as this happens after pull back to $\mathrm{Spec}(K)$). Thus we have a morphism $\rho : G^{\mathrm{der}} \to H^{\mathrm{ad}}$ that (cf. (vi)) extends the central isogeny $\rho_U : G^{\mathrm{der}}_U \to H^{\mathrm{ad}}_U$. From the uniqueness part of Lemma 2.3.1 (a), we get that ρ_U can be identified with the central isogeny $H_U \to H^{\mathrm{ad}}_U$ and thus we can identify $G^{\mathrm{der}}_U = H_U$.

Let $\mathfrak{h} = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Phi} \mathfrak{h}_\alpha$ be the root decomposition of $\mathfrak{h} = \mathrm{Lie}(H)$ with respect to the split, maximal torus T of H. So Φ is a root system of characters of T and each $\mathfrak{h}_\alpha$ is a free A-module of rank 1. We choose a basis Δ of Φ. Let Φ^+ (resp. Φ^-) be the set of roots of Φ that are positive (resp. are negative) with respect to Δ. For $\alpha \in \Phi$, let $\mathbb{G}_{a,\alpha}$ be the $\mathbb{G}_{aY}$ subgroup scheme of H that is normalized by T and that has $\mathfrak{h}_\alpha$ as its Lie algebra (cf. [DG, Vol. III, Exp. XXII, Thm. 1.1]). It is known that the product morphism

$$\omega : \times_{\alpha\in\Phi^+} \mathbb{G}_{a,\alpha} \times_Y T \times_Y \times_{\alpha\in\Phi^-} \mathbb{G}_{a,\alpha} \to H$$

is an open embedding regardless of the orders in which the two products of $\mathbb{G}_{aY}$ group schemes are taken (cf. loc. cit.). Let $\Omega := \mathrm{Im}(\omega)$; it is an open subscheme of H. If A is also a strictly henselian ring, then each point $b \in H(k)$ lifts to a point $a \in H(A)$; let $a\Omega$ be the open subscheme of H that is the left translation of Ω by a.

We consider the following statement $\mathfrak{H}$: we have a closed embedding homomorphism $f : H \to GL_M$ that extends the closed embedding $G^{\mathrm{der}}_K = H_K \hookrightarrow GL_{M\otimes_A K}$. Obviously $\mathfrak{H}$ implies $\mathfrak{S}$. Thus to end the proof of the Theorem it suffices to check that statement $\mathfrak{H}$ is true. For this, we will consider the following three cases on the possible types of A.

5.3.2. The discrete valuation ring case. Suppose that A is a discrete valuation ring. Thus G^{der} is a closed subgroup scheme of GL_M and ρ is a homomorphism. We check that ρ is a quasi-finite morphism whose fibres are surjective. Obviously ρ is of finite type. Thus it suffices to check that the special fibre $\rho_k : G^{\mathrm{der}}_k \to H^{\mathrm{ad}}_k$ of ρ over y is a quasi-finite, surjective morphism. For this we can also assume that A is complete and has a separable closed residue field. Each split, maximal torus of H_k lifts to a split torus of H, cf. Proposition 2.3. From this and (vi) we get that the image of ρ_k contains all split, maximal tori of H^{ad}_k. As such tori generate H^{ad}_k, ρ_k is surjective. We have equalities $\dim(H^{\mathrm{ad}}_k) = \dim(H^{\mathrm{ad}}_K) = \dim(G^{\mathrm{der}}_K) = \dim(G^{\mathrm{der}}_k)$ (the last one as G^{der} is the Zariski closure of G^{der}_K in GL_M). By reasons of dimensions, we get that the surjective homomorphism ρ_k is an isogeny. Thus ρ_k is a quasi-finite, surjective morphism.

As ρ is quasi-finite, the normalization of G^{der} is an open subscheme X^{der} of the normalization of H^{ad} in G_K (cf. Zariski Main Theorem). But this normalization of H^{ad} is H, cf. Lemma 2.3.1 (a). Let $X^{\mathrm{der}} \to GL_M$ be the finite morphism defined by the closed embedding $G \hookrightarrow GL_M$. Obviously X^{der} contains H_K and the generic point of the special fibre of H; thus $H \setminus X^{\mathrm{der}}$ has codimension in H at least 2. The morphism $X^{\mathrm{der}} \to GL_M$ extends to a morphism $f : H \to GL_M$ (cf. [BLR, Ch. 4, 4.4, Thm. 1]) that is a homomorphism as its pull back to $U = \mathrm{Spec}(K)$ is so. As $\mathfrak{h} = \mathrm{Lie}(H)$ is a direct

summand of $\mathfrak{gl}_M$, the fibre f_k of f over y has a kernel whose Lie algebra is trivial. Thus $\mathrm{Ker}(f_k)$ is a finite, étale, normal subgroup of H_k and therefore also of $Z(H_k)$ and of T_k. As T is a torus of GL_M, we get that $\mathrm{Ker}(f_k)$ is the trivial group. Thus the fibres of f are closed embeddings. Thus f is a closed embedding homomorphism, cf. Theorem 2.5. Therefore the statement $\mathfrak{H}$ holds.

5.3.3. The normal case. Suppose that A is local, excellent, and normal. Based on Subsubsection 5.3.2, we can assume A has dimension at least 2. To check the statement $\mathfrak{H}$ we can also assume that A is strictly henselian. We identify $\mathbb{G}_{a,\alpha_U}$ with a closed subgroup scheme of G_U^{der} and thus also of $(GL_M)_U$. The closed embedding homomorphism $\mathbb{G}_{a,\alpha_U} \hookrightarrow (GL_M)_U$ extends to a morphism $\mathbb{G}_{a,\alpha} \to GL_M$ (cf. [BLR, Ch. 4, 4.4, Thm. 1]) that is automatically a homomorphism. Using such homomorphisms and the closed embedding $T \hookrightarrow GL_M$, we get that we have a natural morphism $\Omega \to GL_M$ that extends the (locally closed) embedding $\Omega_U \hookrightarrow (GL_M)_U$ and that is compatible with the closed embedding homomorphism $H_K = G_K^{\mathrm{der}} \hookrightarrow GL_{M\otimes_A K}$. As $H \setminus (\Omega \cup H_K)$ has codimension in H at least 2, from loc. cit. we get that the morphisms $\Omega \to GL_M$ and $H_K \to GL_M$ extend to a morphism $f : H \to GL_M$ that is automatically a homomorphism. As $G_U^{\mathrm{der}} = H_U$, the fibres of f over points of U are closed embeddings. As $\mathfrak{h} = \mathrm{Lie}(H)$ is a direct summand of $\mathfrak{gl}_M$, the fibre f_k of f over y has a kernel whose Lie algebra is trivial. Thus $\mathrm{Ker}(f_k)$ is a finite, étale, normal subgroup of H_k; as in the end of Subsubsection 5.3.2 we argue that $\mathrm{Ker}(f_k)$ is the trivial group. Thus the fibres of f are closed embeddings and therefore f is a closed embedding homomorphism, cf. Theorem 2.5. Thus the statement $\mathfrak{H}$ is true.

5.3.4. The general case. In this Subsubsection we only assume that A is an excellent, local, reduced ring, cf. the beginning of the proof. Thus we will not assume that Y is integral and therefore to check the statement $\mathfrak{H}$ we can assume that A is strictly henselian. Let $Y^{\mathrm{n}} = \mathrm{Spec}(A^{\mathrm{n}})$ be the normalization of Y; we have $K_{Y^{\mathrm{n}}} = K$. The morphism $Y^{\mathrm{n}} \to Y$ is finite, as A is excellent. Based on Subsubsection 5.3.3 applied to the local rings of Y^{n} that dominate Y, we have a natural closed embedding homomorphism $H_{Y^{\mathrm{n}}} \hookrightarrow (GL_M)_{Y^{\mathrm{n}}}$. Thus we also have a natural finite morphism $f^{\mathrm{n}} : H_{Y^{\mathrm{n}}} \to GL_M$ that extends the closed embedding $H_K = G_K^{\mathrm{der}} \hookrightarrow GL_{M\otimes_A K}$. We will show that f^{n} factors as a morphism $f : H \to GL_M$. Let $\mathfrak{h} = \mathrm{Lie}(T) \bigoplus_{\alpha\in\Phi} \mathfrak{h}_\alpha$, $\mathbb{G}_{a,\alpha}$, and Ω be as in Subsubsection 5.3.1.

Let c be $\mathrm{char}(k)-1$ if $\mathrm{char}(k) \in \mathbb{N}^*$ and be ∞ if $\mathrm{char}(k) = 0$. Due to property (iii) (see (vi)), the map $\mathrm{Exp}(u) := \sum_{l=0}^{c} \frac{u^l}{l!}$ defined for $u \in \mathfrak{h}_\alpha$ defines naturally a homomorphism $\mathbb{G}_{a,\alpha} \to GL_M$. Thus as in Subsubsection 5.3.3 we argue that we have a natural morphism $\Omega \to GL_M$. In other words, f^{n} restricted to $\Omega_{Y^{\mathrm{n}}}$ factors as a morphism $\Omega \to GL_M$. The product morphism $\Omega \times_Y \Omega \to H$ is surjective and smooth. So as A is strictly henselian, the product map $\Omega(A) \times \Omega(A) \to H(A)$ is surjective. This and the existence of the morphism $\Omega \to GL_M$ imply that for each point $a \in H(A)$, the restriction of f^{n} to $(a\Omega)_{Y^{\mathrm{n}}}$ factors also through a morphism $a\Omega \to GL_M$. As k is infinite, $H(k)$ is Zariski dense in H_k (cf. [Bo, Cor. 18.3]). Thus H is the union of H_U and of its open subschemes of the form $a\Omega$. This implies the existence of $f : H \to GL_M$. As in the end of Subsubsection 5.3.3 we argue that the fibres of f are closed embeddings and thus that f is a closed embedding. Thus the statement $\mathfrak{H}$ is true. This ends the proof of the basic Theorem B. $\square$

5.4. Remarks on the Basic Theorem B. (a) We can always choose the family $(\tilde{T}_{i\tilde{K}})_{i\in I}$

of tori of condition 5.3 (iv) such that condition 5.3 (iv.a) holds. If $\mathrm{char}(k) = 0$, then conditions 5.3 (iii) and (iv.c) always hold.

Suppose that Y is also normal and G_K is semisimple. Then conditions 5.3 (i), (iii), (iv.b), (iv.d), and (v) hold. Obviously, condition 5.3 (iv.c) holds if Y is a $\mathrm{Spec}(\mathbb{Z}[\frac{1}{\theta}])$-scheme, where $\theta \in \mathbb{N}^*$ is effectively computable in terms of a fixed family $(\tilde{T}_{i\tilde{K}})_{i\in I}$ of tori for which condition 5.3 (iv.a) holds.

(b) Conditions 5.3 (i), (ii), and (iv.b) (resp. 5.3 (iii) and (iv.c)) are (resp. are not) implied by the assumption that G is a reductive subgroup scheme of GL_M. If $\mathrm{char}(K) = 0$, then condition 5.3 (iv.c) implies condition 5.3 (iii). Condition 5.3 (v) holds if Y is strictly henselian, cf. Proposition 2.3.

Condition 5.3 (iv.c) is needed in general. Here is one example. Let p be a prime, let $A = \mathbb{Z}_p$, let $G = SL_{2\mathbb{Z}_p}$, let M_1 be the irreducible left G-module of rank $p+1$ that is the p-th symmetric power of the standard left G-module of rank 2, and let a_p and $a_{-p} \in M_1$ be such that they generate direct summands of M_1 on which a rank 1 split torus T of G acts via the p-th power and the $-p$-th power (respectively) of a fixed isomorphism $T \xrightarrow{\sim} \mathbb{G}_{m\mathbb{Z}_p}$. It is easy to see that the $\mathbb{Z}_p$-module $M := M_1 + \mathbb{Z}_p \frac{a_p + a_{-p}}{p}$ is a left $\mathrm{Lie}(G)$-module. But M_1 is not a left T-module and thus it is also not a left G-module.

(c) Let $p \in \mathbb{N}^*$ be a prime. Let $Y = \mathrm{Spec}(A)$ be an arbitrary local, reduced scheme that is not normal and whose residue field k has characteristic p. Let $K := K_Y$. We have a variant of the Basic Theorem B over $Y = \mathrm{Spec}(A)$. For this variant one only needs to replace conditions 5.3 (iii), (iv), and (v) by the following single condition:

(iii) locally in the étale topology of Y, there exists a maximal torus T of H and there exists a family of split subtori $(T_i)_{i\in I}$ of T which are of rank 1, which are equipped with isomorphisms $i_{T_i} : T_i \to \mathbb{G}_{mY}$, and for which the following three properties hold:

(iii.a) the product homomorphism $\times_{i\in I} T_i \to T$ is an isogeny;

(iii.b) for each $i \in I$ we have a direct sum decomposition $M \otimes_A K = \oplus_{\gamma_i \in \mathcal{S}_i} M_{i\gamma_i K}$ in free K-modules such that T_{iK} acts on $M_{i\gamma_i K}$ via the diagonal character γ_i of T_{iK} with respect to $i_{T_i} \times_Y \mathrm{Spec}(K)$ (thus $\mathcal{S}_i$ is a finite subset of the group of diagonal characters of T_{iK} with respect to $i_{T_{iK}}$, viewed additively and identified with $\mathbb{Z}$);

(iii.c) for each $i \in I$ either:

– the set $\mathcal{S}_i$ is of p-type 1, or

– $\mathcal{S}_i$ is of p-type 2 (resp. of p-type) and T_i is a torus of a semisimple, closed subgroup scheme S_i of H that is isomorphic to SL_{2Y} (resp. to PGL_{2Y});

(iii.d) for each rank 1 free A-submodule $\mathfrak{n}$ of $\mathfrak{h}$ on which T acts via a non-trivial character of T, the order of nilpotency of every endomorphism $u \in \mathfrak{n} \subseteq \mathrm{End}_A(M)$ is at most p and moreover the exponential map $\mathrm{Exp}_{\mathfrak{n}\otimes_A K} : \mathfrak{n} \otimes_A K \to GL_{M\otimes_A K}(K)$ that takes $u \in \mathfrak{n} \otimes_A K$ into $\mathrm{Exp}_{\mathfrak{n}\otimes_A K}(u) := \sum_{l=0}^{p} \frac{u^l}{l!}$, factors through the group of K-valued points of a $\mathbb{G}_{aK}$-subgroup of G_K^{der} normalized by T_K.

The proof of this variant of the Basic Theorem B is the same.

We have the following practical form of the Basic Theorem B.

5.5. Corollary. *Let $p \in \mathbb{N}^*$ be a prime. Let $Y = Spec(A)$ be a local, integral scheme. Let $K := K_Y$ and let k be the residue field of the closed point y of Y. We assume that $char(k) = p$ and that Y is either normal or strictly henselian. Let M be a free A-module of finite rank. Let G_K be a reductive subgroup of $GL_{M\otimes_A K}$ and let $\tilde{T}_K$ be a maximal torus of G_K. Let $\mathfrak{h} := Lie(G_K^{\mathrm{der}}) \cap \mathfrak{gl}_M$, the intersection being taken inside $\mathfrak{gl}_M \otimes_A K$. Let $Z^0(G)$ and G be the Zariski closures of $Z^0(G_K)$ and G_K (respectively) in GL_M. We also assume that the following three conditions hold:*

(i) *$Z^0(G)$ is a closed subgroup scheme of GL_M that is a torus;*

(ii) *the Lie algebra $\mathfrak{h}$ is a direct summand of $\mathfrak{gl}_M$ and the Killing form $\mathcal{K}_\mathfrak{h}$ is perfect;*

(iii) *the torus $\tilde{T}_K$ is generated by cocharacters that act on $M \otimes_A K$ via the trivial and the identical character of $\mathbb{G}_{mK}$.*

Then G is a reductive, closed subgroup scheme of GL_M.

Proof: We check that the conditions 5.3 (i) to (v) hold in our present context. Obviously condition 5.3 (i) holds, cf. (i). From Proposition 3.6 (b) and (ii) we get that $\mathrm{Lie}(G_K^{\mathrm{der}}) = \mathfrak{h}\otimes_A K$ is $\mathrm{Lie}(G_K^{\mathrm{ad}})$. Let H^{ad} be the adjoint group scheme over Y which extends G_K^{ad} and for which we have an identity $\mathrm{Lie}(H^{\mathrm{ad}}) = \mathfrak{h}$ that extends the identities $\mathrm{Lie}(H_K^{\mathrm{ad}}) = \mathrm{Lie}(G_K^{\mathrm{ad}}) = \mathfrak{h} \otimes_A K$, cf. Theorem 1.1 (a) and (ii). The fibres of the central isogeny $H^{\mathrm{sc}} \to H^{\mathrm{ad}}$ are étale isogenies, cf. Proposition 3.6 (b) and (ii). Let $H \to H^{\mathrm{ad}}$ be the central isogeny that extends the central isogeny $G_K^{\mathrm{der}} \to G_K^{\mathrm{ad}} = H_K^{\mathrm{ad}}$ (cf. Lemma 2.3.1 (a)); it is étale. We have $\mathrm{Lie}(H) = \mathrm{Lie}(H^{\mathrm{ad}}) = \mathfrak{h}$ and thus the condition 5.3 (ii) holds. We take $\tilde{K} = K$. Let $(\tilde{T}_{iK})_{i\in I}$ be the family of $\mathbb{G}_{mK}$ subgroups of $\tilde{T}_K$ that are the images of the cocharacters mentioned in (iii). Obviously conditions 5.3 (iv.a), (iv.b), and (iv.c) hold (each set $\mathcal{S}_i$ as in condition 5.3 (iv.c) is a subset of $\{0, 1\}$ and thus it is of p-type 1). As $H \to H^{\mathrm{ad}}$ is an étale cover, we easily get that condition 5.3 (iv.d) holds. Thus condition 5.3 (iv) holds.

The torus $\tilde{T}_K$ is split, cf. (iii). Let $\mathrm{Lie}(G_K) = \mathfrak{h} \otimes_A K = \mathrm{Lie}(\tilde{T}_K) \bigoplus_{\alpha\in\Phi} \mathfrak{h}_{\alpha,K}$ be the root decomposition with respect to the split, maximal torus $\tilde{T}_K$ of G_K. Based on the generation part of (iii), for each $\alpha \in \Phi$ there exists an element $i_\alpha \in I$ such that $\tilde{T}_{i_\alpha K}$ does not fix $\mathfrak{h}_{\alpha,K}$. From the action part of (iii) applied to $\tilde{T}_{i_\alpha K}$, we get that we have $u^2 = 0$ for each $u \in \mathfrak{h}_{\alpha,K} \subseteq \mathfrak{gl}_{M\otimes_A K}$ and that the exponential map $\mathrm{Exp}_{\mathfrak{h}_{\alpha,K}} : \mathfrak{h}_{\alpha,K} \to GL_{M\otimes_A K}(K)$, which takes $u \in \mathfrak{h}_{\alpha,K}$ into $1_{M\otimes_A K} + u$, factors through the group of K-valued points of the $\mathbb{G}_{aK}$-subgroup of G_K^{der} which is normalized by $\tilde{T}_K$ and whose Lie algebra is $\mathfrak{h}_{\alpha,K}$. Thus condition 5.3 (iii) holds even if Y is not normal. If Y is strictly henselian, then condition 5.3 (v) holds even if Y is not normal (cf. the strictly henselian part of Remark 5.4 (b)). Thus condition 5.3 (v) holds. We conclude that all the conditions 5.3 (i) to (v) hold. Therefore the Corollary follows from the Basic Theorem B. □

6. An abstract crystalline application

Let $p \in \mathbb{N}^*$ be a prime. Let k be a perfect field of characteristic p. Let $W(k)$ be the ring of Witt vectors with coefficients in k. Let σ be the Frobenius automorphism of k and $W(k)$. Let $R := W(k)[[x]]$, where x is an independent variable. The p-adic completion of the R-module of differentials Ω_R is free of rank 1 and is naturally identified with Rdx.

Let $\Phi_R : R \to R$ be the Frobenius lift that is compatible with σ and that takes x to x^p. Let $d\Phi_R : Rdx \to Rdx$ be the differential map of Φ_R. Let O be the local ring of R that is a discrete valuation ring of mixed characteristic $(0, p)$. Let O_1 be the p-adic completion of O. Let $\Phi_{O_1} : O_1 \to O_1$ be the Frobenius lift induced by Φ_R. Let K_1 be the field of fractions of O_1. Let $k_1 := O/pO = O_1/pO_1 = k((x))$. Let $Y := \mathrm{Spec}(R)$. Let y be the closed point of Y. Let $U := Y \setminus \{y\}$. We identify y with morphisms $y : \mathrm{Spec}(k) \to Y$ and $y : \mathrm{Spec}(k) \to \mathrm{Spec}(k[[x]])$. Let $y_1 : \mathrm{Spec}(k_1) \to \mathrm{Spec}(k[[x]])$ be the natural dominant morphism. Let $K := K_Y$ be the field of fractions of R. For general properties of F-crystals over k, $k[[x]]$, and k_1 we refer to [Be], [BM], and [Ka].

In this Section we combine results of the previous Sections with de Jong's extension theorem to obtain a new and meaningful crystalline result that pertains to specializations of affine group schemes in crystalline contexts (see the Basic Theorem C of 6.3). In Subsections 6.1 and 6.2 we introduce the setting and the axioms (respectively) needed for the proof of the Basic Theorem C. In Remarks 6.4 we compare the range of applicability of the Basic Theorems B and C to contexts that pertain to integral models of Shimura varieties of Hodge type.

6.1. A setting. Let $\mathfrak{C}$ be an F-crystal over $k[[x]] = R/pR$. We recall that $\mathfrak{C}$ is uniquely determined by a triple (M, Φ, ∇), where M is a free R-module of finite rank, $\Phi : M \to M$ is a Φ_R-linear endomorphism, and $\nabla : M \to M \otimes_R Rdx$ is a topologically nilpotent (automatically integrable) connection on M, such that the following two properties hold: (i) we have $\nabla \circ \Phi = (\Phi \otimes d\Phi_R) \circ \nabla$, and (ii) there exists $q \in \mathbb{N}$ such that $p^q M$ is contained in the R-span of $\Phi(M)$. Let $\mathfrak{C}_{y_1}$ be the F-crystal over k_1 that is the pull back of $\mathfrak{C}$ via $y_1 : \mathrm{Spec}(k_1) \to \mathrm{Spec}(k[[x]])$. As $\{x\}$ is a finite p-basis of k_1 in the sense of [BM, Def. 1.1.1], from [BM, Prop. 1.3.3] we get that $\mathfrak{C}_{y_1}$ is uniquely determined by the triple $(M \otimes_R O_1, \Phi \otimes \Phi_{O_1}, \nabla_1)$; here ∇_1 is the extension of ∇ to a connection on $M \otimes_R O_1$.

For $i \in \mathbb{N}^*$ and for an F-crystal $\mathfrak{B}$ over k_1 (or over $k[[x]]$), let $\mathfrak{B}^{\otimes i}$ be the F-crystal over k_1 (or over $k[[x]]$) that is the tensor product of i copies of $\mathfrak{B}$. For a commutative R-algebra A, let $\mathcal{T}_A$ be the trace form on $\mathrm{End}_A(M \otimes_R A)$.

6.2. The axioms. Let $s \in \mathbb{N}^*$. Let $\mathfrak{D}_{y_1} := \oplus_{i=1}^s \mathfrak{C}_{y_1}^{\otimes i}$. Let $\tilde{M} := \oplus_{i=1}^s M^{\otimes i}$. Let $\tilde{M}_1 := \tilde{M} \otimes_R O_1$. Let $(t_\alpha)_{\alpha \in \mathcal{J}}$ be a finite family of endomorphisms of $\mathfrak{D}_{y_1}$. Based on [BM, Lemma 3.1.1], we can identify each t_α with an element of $\mathrm{End}_{O_1}(\tilde{M}_1)$. Let E_{K_1} be the subgroup of $GL_{M \otimes_R K_1}$ that fixes t_α for all $\alpha \in \mathcal{J}$. Let G_{K_1} be a connected, normal subgroup of E_{K_1}. We list six axioms on the family of endomorphisms $(t_\alpha)_{\alpha \in \mathcal{J}}$ and on the subgroup G_{K_1} of E_{K_1}:

(i) the normalization G_{O_1} of the Zariski closure of G_{K_1} in $GL_{M \otimes_R O_1}$, is a reductive group scheme over $\mathrm{Spec}(O_1)$;

(ii) there exists a subset $\mathcal{J}_{\mathrm{c}}$ of $\mathcal{J}$ such that the following three properties hold:

(ii.a) the set $\{t_\alpha | \alpha \in \mathcal{J}_{\mathrm{c}}\}$ is a semisimple $\mathbb{Z}_p$-subalgebra $\mathcal{A}$ of $\mathrm{End}_{O_1}(M \otimes_R O_1)$;

(ii.b) the center of the centralizer of $\mathcal{A} \otimes_{\mathbb{Z}_p} O_1$ in $GL_{M \otimes_R O_1}$ is a torus T_{O_1} of $GL_{M \otimes_R O_1}$ that has $Z^0(G_{O_1})$ as a subtorus;

(ii.c) there exists a subset $\mathcal{J}_{\mathrm{c}}^0$ of $\mathcal{J}_{\mathrm{c}}$ such that $\{t_\alpha | \alpha \in \mathcal{J}_{\mathrm{c}}^0\}$ is an O_1-basis for $\mathrm{Lie}(Z^0(G_{O_1}))$;

(iii) there exists a subset $\mathfrak{J}_{\rm d}$ of $\mathfrak{J}$ with the property that for each $\alpha \in \mathfrak{J}_{\rm d}$, there exists $n(\alpha) \in \mathbb{Z}$ such that $p^{n(\alpha)} t_\alpha \in \mathrm{End}_{K_1}(M^{\otimes 2} \otimes_R K_1) = \mathrm{End}_{K_1}(\mathrm{End}_{K_1}(M \otimes_R K_1))$ is a projector of $\mathrm{End}_{K_1}(M \otimes_R K_1)$ whose image is the Lie algebra of a normal, semisimple subgroup $G^\alpha_{K_1}$ of $G^{\rm der}_{K_1}$ of some isotypic Dynkin type;

(iv) for each $\alpha \in \mathfrak{J}_{\rm d}$, the kernel $\mathrm{Ker}(p^{n(\alpha)} t_\alpha)$ is the perpendicular on $\mathrm{Lie}(G^\alpha_{K_1})$ with respect to the trace form $\mathcal{T}_{K_1}$ and moreover the Killing form $\mathcal{K}_{\mathrm{Lie}(G^\alpha_{K_1})}$ is a non-zero rational multiple of the restriction of $\mathcal{T}_{K_1}$ to $\mathrm{Lie}(G^\alpha_{K_1})$;

(v) we have a natural isogeny $\iota_1 : \times_{\alpha \in \mathfrak{J}_{\rm d}} G^\alpha_{K_1} \to G^{\rm der}_{K_1}$;

(vi) if $p = 2$, then the group G_{K_1} has no normal subgroup that is adjoint of isotypic B_n Dynkin type for some $n \in \mathbb{N}^*$.

6.3. Basic Theorem C (the crystalline reductive extension principle). *Suppose that the family $(t_\alpha)_{\alpha \in \mathfrak{J}}$ of endomorphisms of $\mathfrak{D}_{y_1}$ and the subgroup G_{K_1} of E_{K_1} are such that the axioms 6.2 (i) to (v) hold. Then the following two things hold:*

(a) *the group G_{K_1} is the pull back of a unique closed subgroup G_K of $GL_{M \otimes_R K}$ and moreover the normalization G of the Zariski closure of G_K in GL_M is a reductive group scheme over Y for which we have a finite homomorphism $G \to GL_M$;*

(b) *if the axiom 6.2 (vi) also holds, then G of (a) is in fact a reductive, closed subgroup scheme of GL_M.*

Proof: Let $\mathfrak{D} := \oplus_{i=1}^s \mathfrak{C}^{\otimes i}$. A fundamental result of de Jong (see [dJ, Thm. 1.1]) asserts that each t_α extends to an endomorphism of $\mathfrak{D}$; thus we have $t_\alpha \in \mathrm{End}_R(\tilde{M}) \subseteq \mathrm{End}_{O_1}(\tilde{M}_1)$ for all $\alpha \in \mathfrak{J}$. Therefore we can speak about the subgroup E_K of $GL_{M \otimes_R K}$ that fixes t_α for all $\alpha \in \mathfrak{J}$. The notations match i.e., the group E_{K_1} of Subsection 6.2 is the pull back of E_K to $\mathrm{Spec}(K_1)$. We first show that G_K exists and thus that G is well defined.

6.3.1. Constructing G_K. We check (resp. for $\alpha \in \mathfrak{J}_{\rm d}$ we check) that there exists a unique connected subgroup $G^{\rm der}_K$ (resp. G^α_K) of E_K whose Lie algebra is the Lie subalgebra

$$\mathfrak{l}_K := \oplus_{\alpha \in \mathfrak{J}_{\rm d}} p^{n(\alpha)} t_\alpha(\mathrm{End}_K(M \otimes_R K)) \;\; (\text{resp. } \mathfrak{l}^\alpha_K := p^{n(\alpha)} t_\alpha(\mathrm{End}_K(M \otimes_R K)))$$

of $\mathrm{Lie}(E_K)$. The uniqueness part is implied by [Bo, Ch. I, Subsection 7.1]. To check the existence of $G^{\rm der}_K$ (resp. of G^α_K) we can assume that the set $\mathfrak{J}_{\rm d}$ is not empty. As $\mathfrak{l}_K \otimes_K K_1 = \oplus_{\alpha \in \mathfrak{J}_{\rm d}} \mathfrak{l}^\alpha_K \otimes_K K_1 = \oplus_{\alpha \in \mathfrak{J}_{\rm d}} \mathrm{Lie}(E^\alpha_{K_1})$ is a semisimple Lie algebra over K_1, both $\mathfrak{l}_K$ and $\mathfrak{l}^\alpha_K$ are semisimple Lie algebras over K. Therefore we have $\mathfrak{l}_K = [\mathfrak{l}_K, \mathfrak{l}_K]$ and $\mathfrak{l}^\alpha_K = [\mathfrak{l}^\alpha_K, \mathfrak{l}^\alpha_K]$. From this and [Bo, Ch. I, Corollary 7.9] we get that there exists a unique connected subgroup $G^{\rm der}_K$ (resp. G^α_K) of $GL_{M \otimes_R K}$ whose Lie algebra is $\mathfrak{l}_K$ (resp. is $\mathfrak{l}^\alpha_K$). From [Bo, Ch. I, Subsection 7.1] we get that $G^{\rm der}_K$ is a subgroup of E_K and that G^α_K is a subgroup of $G^{\rm der}_K$. Thus $G^{\rm der}_K$ and G^α_K exist.

The pull back of $G^{\rm der}_K$ (resp. of G^α_K) to $\mathrm{Spec}(K_1)$ is a connected subgroup of E_{K_1} that has the same Lie algebra as the derived group $G^{\rm der}_{K_1}$ (resp. as the normal, semisimple subgroup $G^\alpha_{K_1}$) of G_{K_1}, cf. axiom 6.2 (v) (resp. cf. the definition of $\mathfrak{l}^\alpha_K$). From [Bo, Ch. I, Subsection 7.1] we get that this pull back is $G^{\rm der}_{K_1}$ (resp. is $G^\alpha_{K_1}$). Thus our notations

match and we have an isogeny $\iota : \times_{\alpha\in\mathcal{J}_{\mathrm{d}}} G_K^\alpha \to G_K^{\mathrm{der}}$ of semisimple groups over $\mathrm{Spec}(K)$ whose pull back to $\mathrm{Spec}(K_1)$ is the isogeny ι_1 of the axiom 6.2 (v).

As we have $t_\alpha \in \mathrm{End}_R(M)$ for all $\alpha \in \mathcal{J}_{\mathrm{c}}$, we can identify $\mathcal{A}$ of the axiom 6.2 (ii) with a semisimple $\mathbb{Z}_p$-subalgebra of $\mathrm{End}_R(M)$. As $\mathcal{A}$ is formed by elements of $\mathrm{End}_R(M)$ that commute with Φ, the semisimple R-algebra $\mathcal{A}\otimes_{\mathbb{Z}_p} R$ is an R-subalgebra of $\mathrm{End}_R(M)$. Thus the centralizer of $\mathcal{A}\otimes_{\mathbb{Z}_p} R$ in $\mathrm{End}_R(M)$ is a semisimple R-algebra. This implies that the center of the centralizer of $\mathcal{A}\otimes_{\mathbb{Z}_p} R$ in GL_M is a torus T whose pull back to $\mathrm{Spec}(O_1)$ is the torus T_{O_1} of the axiom 6.2 (ii.b).

We check that there exist a unique subtorus $Z^0(G_K)$ of T_K whose Lie algebra $\mathfrak{t}$ is K-generated by $\{t_\alpha|\alpha\in\mathcal{J}_{\mathrm{c}}^0\}$. Let $\tilde{K}$ be a finite Galois extension of K such that $T_{\tilde{K}}$ is a split torus. Let $\tilde{K}_1$ be the composite field of $\tilde{K}$ and K_1. Let $Z^0(G_{\tilde{K}})$ be the unique subtorus of $T_{\tilde{K}}$ whose extension to $\tilde{K}_1$ is $Z^0(G_{1K_1})\times_{\mathrm{Spec}(K_1)}\tilde{K}_1$. We have $\mathrm{Lie}(Z^0(G_{\tilde{K}}))\otimes_{\tilde{K}}\tilde{K}_1 = \mathfrak{t}\otimes_K \tilde{K}_1$ and thus $\mathrm{Lie}(Z^0(G_{\tilde{K}})) = \mathfrak{t}\otimes_K\tilde{K}$. The subtorus $Z^0(G_{\tilde{K}})$ of $T_{\tilde{K}}$ is the unique subtorus whose Lie algebra is $\mathfrak{t}\otimes_K\tilde{K}$, cf. [Bo, Ch. I, Subsection 7.1]. Based on this uniqueness the Galois group $\mathrm{Gal}(\tilde{K}/K)$ acts naturally and freely on the subtorus $Z^0(G_{\tilde{K}})$ of $T_{\tilde{K}}$. The quotient of this Galois action exists (cf. [BLR, Ch. 6, 6.1, Thm. 6]) and it is the unique subtorus $Z^0(G_K)$ of T_K whose Lie algebra is $\mathfrak{t}$.

As $\mathfrak{t}$ and $\mathrm{Lie}(G_K^{\mathrm{der}})$ commute (as they commute after tensorization with K_1), the groups $Z^0(G_K)$ and G_K^{der} also commute (cf. [Bo, Ch. I, 7.1, Prop. 7.8]). Thus the subgroup G_K of E_K generated by G_K^{der} and $Z^0(G_K)$ is reductive and the notations match. It is easy to see that the pull back of G_K to $\mathrm{Spec}(K_1)$ is the subgroup G_{K_1} of E_{K_1}. As G_K exists, the group scheme G is well defined.

6.3.2. Applying 5.1. We recall that $G_U := G\times_Y U$ and $G_O := G\times_Y \mathrm{Spec}(O)$, cf. Subsection 2.1. Thus G_U is a flat, affine group scheme over U. Based on Proposition 5.1 (a), to prove (a) we only have to check that G_U is a reductive group scheme over U. As $W(k)$ is an excellent ring (cf. [Ma, §34]), the morphism $\mathrm{Spec}(O_1)\to\mathrm{Spec}(O)$ is regular. Thus the morphism $G_O\times_{\mathrm{Spec}(O)}\mathrm{Spec}(O_1)\to G_O$ is also regular, cf. [Ma, (33.D), Lemma 4]. Therefore $G_O\times_{\mathrm{Spec}(O)}\mathrm{Spec}(O_1)$ is a normal scheme (cf. [Ma, (33.B), Lemma 2]) which is finite over $GL_{M\otimes_O O_1}$. Thus we can identify $G_O\times_{\mathrm{Spec}(O)}\mathrm{Spec}(O_1)$ with G_{O_1}. From this and the axiom 6.2 (i), we get that G_O is a reductive group scheme over $\mathrm{Spec}(O)$. Let $V\in\mathcal{D}(Y)\backslash\{O\}$; it is a local ring of Y that is a discrete valuation ring of equal characteristic 0. The normalization of the Zariski closure of G_K in $GL_{M\otimes_R V}$ is $G_V = G\times_Y V$. To end the argument that G_U is a reductive group scheme over U (i.e., to end the proof of (a)), we only have to check that for each $V\in\mathcal{D}(Y)\setminus\{O\}$ the group scheme G_V is reductive; we will show that in fact G_V is a reductive, closed subgroup scheme of $GL_{M\otimes_R V}$.

6.3.3. Local study at V. Let $V\in\mathcal{D}(Y)\setminus\{O\}$; it is a local ring of Y that is a discrete valuation ring of equal characteristic 0. Let $Z^0(G_V)$ be the Zariski closure of $Z^0(G_K)$ in $GL_{M\otimes_R V}$. It is a subtorus of T_V (as one can easily check this over the spectrum of the strict henselization of V). For $\alpha\in\mathcal{J}_{\mathrm{d}}$ let G_V^α be the Zariski closure of G_K^α in $GL_{M\otimes_R V}$. Let $\mathfrak{l}_V^\alpha$ be the image of $\mathrm{End}_V(M\otimes_R V)$ through $p^{n(\alpha)}t_\alpha$ (equivalently through t_α). The element $p^{n(\alpha)}t_\alpha$ is a projector of $\mathrm{End}_V(M\otimes_R V)$, as this is so after tensorization with K_1 over V (cf. axiom 6.2 (iii)). Thus $\mathfrak{l}_V^\alpha$, as a V-submodule of $\mathrm{End}_V(M\otimes_R V)$, is a direct summand. Moreover we have $\mathfrak{l}_V^\alpha\otimes_V K = \mathfrak{l}_K^\alpha = \mathrm{Lie}(G_K^\alpha)$. Therefore $\mathfrak{l}_V^\alpha$ is a Lie subalgebra

of $\mathrm{End}_V(M \otimes_R V)$. Due to the axiom 6.2 (iv), $\mathrm{Ker}(p^{n(\alpha)} t_\alpha)$ is the perpendicular on $\mathfrak{l}_V^\alpha$ with respect to $\mathcal{T}_V$. Thus the restriction $\mathcal{T}_V^\alpha$ of the trace form $\mathcal{T}_V$ to $\mathfrak{l}_V^\alpha$ is perfect.

The Killing form $\mathcal{K}_{\mathfrak{l}_V^\alpha}$ is a multiple of $\mathcal{T}_V^\alpha$ by a non-zero rational number (as this is so after tensorization with K_1 over V, cf. axiom 6.2 (iv)). Thus $\mathcal{K}_{\mathfrak{l}_V^\alpha}$ is a multiple of $\mathcal{T}_V^\alpha$ by an invertible element of V. Thus $\mathcal{K}_{\mathfrak{l}_V^\alpha}$ is perfect. Let $G_V^{\alpha \mathrm{ad}}$ be the adjoint group scheme over $\mathrm{Spec}(V)$ that extends $G_K^{\alpha \mathrm{ad}}$ and that has the property that the identities $\mathrm{Lie}(G_K^{\alpha \mathrm{ad}}) = \mathrm{Lie}(G_K^\alpha) = \mathfrak{l}_V^\alpha \otimes_V K$ extend to an identity $\mathrm{Lie}(G_V^{\alpha \mathrm{ad}}) = \mathfrak{l}_V^\alpha$, cf. Theorem 1.1 (a). The normalization $\tilde{G}_V^\alpha$ of $G_V^{\alpha \mathrm{ad}}$ in G_K^α is a semisimple group scheme over $\mathrm{Spec}(V)$ (cf. Lemma 2.3.1 (b)) whose Lie algebra is $\mathfrak{l}_V^\alpha$. From the Basic Theorem B (applied with $(Y, K, G_K, H) = (\mathrm{Spec}(V), K, G_K^\alpha, \tilde{G}_V^\alpha))$, we get that G_V^α is a semisimple, closed subgroup scheme of $GL_{M \otimes_R V}$ which can be identified with $\tilde{G}_V^\alpha$. [Argument: in the context of $(\mathrm{Spec}(V), K, G_K^\alpha, \tilde{G}_V^\alpha)$, the existence of $\tilde{G}_V^\alpha$ implies that condition 5.3 (ii) holds while from Remark 5.4 (a) applied to the regular, integral $\mathrm{Spec}(\mathbb{Q})$-scheme $\mathrm{Spec}(V)$ we get that conditions 5.3 (i), (iii), (iv), and (v) hold.]

6.3.4. End of the proof. The natural Lie homomorphism $\mathrm{Lie}(Z^0(G_K)) \bigoplus_{\alpha \in \mathcal{J}_{\mathrm{d}}} \mathfrak{l}_K^\alpha \to \mathrm{Lie}(G_K)$ is a Lie isomorphism. Thus the reductive, normal subgroups $Z^0(G_K)$ and G_K^α's (with $\alpha \in \mathcal{J}_{\mathrm{d}}$) of G_K commute pairwise, cf. [Bo, Ch. I, 7.1, Prop. 7.8]. For $V \in \mathcal{D}(Y) \setminus \{O\}$ the group schemes $Z^0(G_V)$ and G_V^α's (with $\alpha \in \mathcal{J}_{\mathrm{d}}$) over $\mathrm{Spec}(V)$ are reductive (cf. Subsubsection 6.3.3). Thus G_V is a reductive, closed subgroup of $GL_{M \otimes_R V}$, cf. Proposition 5.2. Thus G_U is a reductive group scheme over U. Thus (a) holds, cf. Subsubsection 6.3.2.

We prove Theorem 6.3 (b). If the axiom 6.2 (vi) also holds, then G_K has no normal subgroup that is adjoint of isotypic B_n Dynkin type with $n \in \mathbb{N}^*$. Thus G is a closed subgroup scheme of GL_M, cf. Proposition 5.1 (b). Thus Theorem 6.3 (b) holds. $\square$

6.4. Remarks. (a) To apply the Basic Theorem C to integral canonical models of Shimura varieties of Hodge type, one works in a context in which the following four things hold:

- the field k is $\bar{\mathbb{F}}_p$ and $\mathfrak{C}$ is the F-crystal over $k[[x]]$ of an abelian scheme A,
- there exists an abelian scheme B over $\mathrm{Spec}(O_1)$ that lifts A_{k_1} and such that for each $\alpha \in \mathcal{J}$, each t_α is a crystalline cycle on B_{K_1}, and
- the group E_{K_1} is a form of the extension to $\mathrm{Spec}(K_1)$ of the Mumford–Tate group F (over $\mathrm{Spec}(\mathbb{Q})$) of a (any) pull back of B to a complex abelian variety (thus E_{K_1} is the reductive subgroup of $GL_{M \otimes_R K_1}$ which is the natural crystalline realization of F);
- the group G_{K_1} corresponds via Fontaine comparison theory to a suitable connected, normal subgroup of $F_{\mathbb{Q}_p}$.

In such a context the axiom 6.2 (vi) is superfluous and the axioms 6.2 (iii) to (v) always hold. If $p > 2$, then axioms 6.2 (i) and (ii) hold if and only if their analogues for the étale cohomology of B_{K_1} with $\mathbb{Z}_p$-coefficients hold (cf. [Va6, Thm. 1.2 and Cor. 1.3 (a)]); moreover, loc. cit. provides weaker forms of this result for $p = 2$. We emphasize that (as [Va6, Subsection 5.5] points out) in the last sentence, it is essential that B is over $\mathrm{Spec}(O_1)$ and not only over the spectrum of a totally ramified, discrete valuation ring extension of O_1 which has residue field k_1 and which is different from O_1.

(b) Suppose that $k = \bar{k}$. Let V be a finite, totally ramified, discrete valuation ring extension of $W(k)$. Let $e := [V : W(k)] \in \mathbb{N}^*$. Let R_e be the p-adic completion of the R-subalgebra of $W(k)[\frac{1}{p}][[x]]$ generated by $\frac{x^{ea}}{a!}$, with $a \in \mathbb{N}^*$; it is a $W(k)$-subalgebra of $W(k)[\frac{1}{p}][[x]]$ and thus it is an integral domain. The k-algebra R_e/pR_e is the inductive limit of its local, artinian k-subalgebras. This implies that R_e is a local, strictly henselian ring whose residue field is k. Let R_e^{n} be the normalization of R_e. We have a natural $W(k)$-epimorphism $R_e^{\mathrm{n}} \twoheadrightarrow W(k)$ that takes x into 0. We also have a $W(k)$-epimorphism $q_\pi : R_e \twoheadrightarrow V$ that takes x into an uniformizer π of V (see [Va6, Subsection 2.2]). Thus we also have a natural $W(k)$-epimorphism $q_\pi^{\mathrm{n}} : R_e^{\mathrm{n}} \twoheadrightarrow V_{\mathrm{big}}$ that extends q_π, where V_{big} is a suitable integral, ind-finite V-algebra. Let $R_e^{\mathrm{n},PD}$ be the p-adic completion of the divided power hull of $\mathrm{Ker}(q_\pi^{\mathrm{n}})$. One can redo [Va1, Subsections 5.2 and 5.3] entirely in the context of R_e^{n} and $R_e^{\mathrm{n},PD}$. This and Subsubsection 5.3.3 applied with $Y = \mathrm{Spec}(R_e^{\mathrm{n}})$ point (resp. the proof of Corollary 5.5 applied with $Y = \mathrm{Spec}(R_e)$ points) out that one can get major shortcuts to [Va1, Prop. 4.3.10 b)] and its crystalline applications of [Va1, Subsections 5.2 and 5.3]. We also note:

$\bullet_-$ that Corollary 5.5 excludes completely the prime $p = 2$ (cf. Corollary 3.8) and that [Va1, Subsections 5.2 and 5.3] require $p \geq 5$;

$\bullet_+$ that the Basic Theorem C also holds for $p \in \{2, 3\}$ and that (a) has many applications even for $p = 2$ (see already math.NT/0311042).

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